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# An operator-valued $Tb$ theorem

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**Abstract**

We establish conditions similar to the $Tb$ theorem of David, Journé and Semmes which guarantee the boundedness of an integral transformation $T$ with $\mathcal{L}(X)$-valued kernel on $L^p_X(\mathbf{R}^n)$, where $1 < p < \infty$ and $X$ is a Banach space with the unconditionality property of martingale differences (UMD).


*Keywords:* Calderón–Zygmund operator; $Tb$ theorem; UMD space; Operator-valued kernel; $R$-boundedness

## 1. Introduction

The purpose of this paper is to provide a version of the celebrated $Tb$ theorem of G. David, J.-L. Journé and S. Semmes [9] for vector-valued functions on $\mathbf{R}^n$ and operator-valued kernels. Before stating the precise result, let us outline some earlier developments into this direction which inspired the present work, and at the same time, our approach to the problem. It is in essence a combination of the ideas and techniques from the three references [7,10,11]—supplied with the recent $R$-boundedness methods [5,20] for treating operator-valued kernels. In [7] G. David presents another proof, due to R. Coifman and S. Semmes, of the $Tb$ theorem in the original scalar-valued setting, whereas [10,11] contain T. Figiel's extension of David and Journé's original $T1$ theorem [8] to the case of

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vector-valued functions and scalar-valued kernels. Here, as usual in the business of singular integrals, "vector-valued" refers to Banach spaces with the UMD property (cf. [4,18]).

The approach of Figiel to the $T1$ theorem is based on the ingenious observation that all Calderón–Zygmund operators (understood in a rather wide sense) on $\mathbf{R}^n$ can be decomposed into sums and products of the following four basic types of operators and their adjoints:

(1) *Haar multipliers*, which map $\chi_D^\epsilon \mapsto \lambda_D^\epsilon \chi_D^\epsilon$, where $\chi_D^\epsilon$ is one of the $2^n - 1$ Haar functions associated with the dyadic cube $D$, and $\lambda_D^\epsilon$ are some numbers;
(2) *Figiel's $T$-operators*, which map $\chi_D^\epsilon \mapsto \chi_{\phi(D)}^\eta$, where $\phi$ is a size-preserving permutation of the dyadic cubes;
(3) *Figiel's $U$-operators*, which map $\chi_D^\epsilon \mapsto \chi_{\phi(D)}^0 - \chi_D^0$, where $\chi_D^0$ is the $L^2$ normalized indicator function of $D$; and
(4) *paraproduct operators*.

This decomposition provides interesting insights into the nature of integral operators even in the scalar-valued setting, but its greatest value lies in the fact that all the elementary mappings of the four types can be related to martingale transforms. This, of course, is an invaluable property when willing to derive estimates for the norms of these operators on spaces of UMD-valued functions, where martingale inequalities constitute the principal analytic tool at our disposal. The relation of Haar multipliers to martingale transforms is particularly clear and simple, whereas the classes of Figiel's $T$ and $U$ operators were introduced and investigated in [10]. The estimates obtained for these operators were then systematically exploited in [11], and combined with a martingale approach to paraproducts, to produce a proof of the $T1$ theorem.

Figiel's approach would be perfect enough for an operator-valued $T1$ theorem, too, at least for the case $T1 = T'1 = 0$; in fact, one could quite readily build a proof of such a result on his work with the Haar system by incorporating the recent $R$-boundedness methods to deal with operator-valued Haar multipliers. There is also a rather different, Fourier-analytic approach to such a theorem, recently devised by L. Weis and the present author [15].

However, neither of these frameworks is very well suited as such when we want to replace the "1" in $T1$ by a general para-accretive function $b$. Figiel's approach, however, appears to be more readily modified to serve this purpose. Thus we adopt from [7] the idea of constructing a new basis of our own, tailor-made for each particular $b$ we might wish to pick, and replace the Haar system in Figiel's proof by this new set of functions. Then it turns out that an "arbitrary" operator can again be decomposed into the analogues of the four types of elementary operators, where one simply replaces the Haar functions by members of the new basis. The idea behind our approach is no more difficult than what was just described; but unfortunately, the proof abounds in technical complications, and is consequently rather lengthy. It should be noted that Figiel did point out in [11] already that his approach could be adapted to more general systems of functions, but at least to the best knowledge of the author, this program was never carried out in detail.

We now formulate (a version of) our main theorem. The assumptions below are not the most general possible, and we will later give a more abstract formulation, similar to

Figiel's $T1$ theorem in [11]. We denote by $\mathcal{R}(\mathcal{T})$ the $R$-bound of the set $\mathcal{T}$ (see [5,20] for related facts).

**Theorem 1.1.** *Let $X$ and $Y$ be UMD spaces, and let the kernel $k$: $\mathbf{R}^n \times \mathbf{R}^n \setminus \{(x,x)\colon x \in \mathbf{R}^n\} \to \mathcal{L}(X,Y)$ satisfy the "standard $R$-estimates"*

$$\mathcal{R}\big(\big\{|x-y|^n k(x,y)\colon x \neq y\big\}\big) < \infty,$$
$$\mathcal{R}\left(\left\{|x-y|^n \log^2\left(\frac{|x-y|}{|x-z|}\right)\big[k(x,y)-k(z,y)\big]\colon 0 < \frac{|x-z|}{|x-y|} < \frac{1}{2}\right\}\right) < \infty,$$
$$\mathcal{R}\left(\left\{|x-y|^n \log^2\left(\frac{|x-y|}{|y-z|}\right)\big[k(x,y)-k(x,z)\big]\colon 0 < \frac{|y-z|}{|x-y|} < \frac{1}{2}\right\}\right) < \infty.$$

*Let $b, \tilde{b} \in L^\infty(\mathbf{R}^n)$ be two para-accretive functions, and let $\mathfrak{t}$ be a bi-linear map from $\mathcal{D}(\mathbf{R}^n) \times \mathcal{D}(\mathbf{R}^n)$ to $\mathcal{L}(X,Y)$ such that*

$$\mathfrak{t}(\phi,\psi) = \iint\limits_{\mathbf{R}^n \times \mathbf{R}^n} \psi(x)\tilde{b}(x)k(x,y)b(y)\phi(y)\,\mathrm{d}x\,\mathrm{d}y,$$

*whenever $\phi, \psi \in \mathcal{D}(\mathbf{R}^n)$ have disjoint supports.*

*Denote $\mathcal{A}_h^t f := t^{n/2} f(t\cdot - h)$. We assume that $\mathfrak{t}$ satisfies the "weak $R$-boundedness property"*

$$\mathcal{R}\big(\big\{\mathfrak{t}\big(\mathcal{A}_h^t\phi, \mathcal{A}_h^t\psi\big)\colon t > 0,\ h \in \mathbf{R}^n;\ \phi,\psi \in \mathcal{B}\big\}\big) < \infty \tag{1.2}$$

*whenever $\mathcal{B}$ is a bounded subset of $\mathcal{D}(\mathbf{R}^n)$.*

*For $\psi \in \mathcal{D}(\mathbf{R}^n)$ satisfying $\int \psi(x)\tilde{b}(x)\,\mathrm{d}x = 0$, define*

$$\mathfrak{t}(1,\psi) := \mathfrak{t}(\chi,\psi) + \iint\limits_{\mathbf{R}^n \times \mathbf{R}^n} \psi(x)\tilde{b}(x)\big[k(x,y)-k(x,z)\big]b(y)\big(1-\chi(y)\big)\,\mathrm{d}x\,\mathrm{d}y,$$

*where $\chi \in \mathcal{D}(\mathbf{R}^n)$ equals $1$ in a neighbourhood of $\operatorname{supp}\psi \ni z$; the specific choice of $\chi$ and $z$ does not alter the definition of $\mathfrak{t}(1,\psi)$. We assume that for all such $\psi$,*

$$\mathfrak{t}(1,\psi) = \int \tilde{w}(x)\tilde{b}(x)\psi(x)\,\mathrm{d}x, \quad \textit{where } \tilde{w} \in BMO_U\big(\mathbf{R}^n\big), \tag{1.3}$$

*and $U \hookrightarrow \mathcal{L}(X,Y)$ is a UMD Banach space, which is also an "$R$-space"; by this we mean that the unit ball $B_U$ is an $R$-bounded subset of $\mathcal{L}(X,Y)$.*

*Defining analogously $\mathfrak{t}(\phi,1)$ for all $\phi \in \mathcal{D}(\mathbf{R}^n)$ with $\int \phi(x)b(x)\,\mathrm{d}x = 0$, we further assume that*

$$\mathfrak{t}(\phi,1)' = \int w(x)b(x)\phi(x)\,\mathrm{d}x, \quad \textit{where } w \in BMO_V\big(\mathbf{R}^n\big), \tag{1.4}$$

*and* $V \hookrightarrow \mathcal{L}(Y', X')$ *is again a UMD R-space.* ((1.3) *and* (1.4) *are the conditions* "$Tb \in BMO$" *and* "$T'\tilde{b} \in BMO$.")

*Then there exists a unique*

$$T \in \bigcap_{1<p<\infty} \mathcal{L}\big(L^p_X(\mathbf{R}^n), L^p_Y(\mathbf{R}^n)\big)$$

*such that* $\langle y' \otimes \tilde{b}\psi, T(x \otimes b\phi)\rangle = \langle y', \mathfrak{t}(\phi, \psi)x\rangle$ *for all* $\phi, \psi \in \mathcal{D}(\mathbf{R}^n)$*, all* $x \in X$ *and* $y' \in Y'$.

Let us have a look at the conditions of Theorem 1.1 for a scalar-valued kernel (or one taking values in a one-dimensional subspace of $\mathcal{L}(X, Y)$). Then the $R$-bounds appearing in the standard $R$-estimates and (1.2) are just uniform bounds, and, moreover, a one-dimensional operator space is both a UMD space and an $R$-space quite obviously, so that all the assumptions reduce to their classical analogues. Actually, the second and third condition in the standard estimates are usually given a somewhat stronger formulation with the function $\log^2$ replaced by $u \mapsto u^\gamma$, $0 < \gamma \leqslant 1$, but the milder logarithmic continuity works just as well. Note that our standard estimates still imply the Hörmander integral condition for $k$, so that an operator $T$ with kernel $k$ satisfying our standard estimates (even with uniform bounds in place of $R$-bounds), once bounded from $L^p_X(\mathbf{R}^n)$ to $L^p_Y(\mathbf{R}^n)$ for some $1 < p < \infty$, is actually bounded for all $1 < p < \infty$ and from $H^1_X(\mathbf{R}^n)$ to $L^1_Y(\mathbf{R}^n)$ and from $L^\infty_X(\mathbf{R}^n)$ to $BMO_Y(\mathbf{R}^n)$. Thus it is easily seen in the case of a scalar kernel, like in the classical $Tb$ theorem, that under the assumption of the standard $R$-estimates, the other assumptions are also necessary for the conclusion of Theorem 1.1.

Actually the scalar-kernel case of Theorem 1.1 (at least with the usual Hölder continuity assumptions) is an immediate corollary of the classical $T1$ and $Tb$ theorems plus Figiel's $T1$ theorem: Let an operator $T$ with scalar-valued kernel $k$ satisfy the assumptions of Theorem 1.1. Then by the David–Journé–Semmes theorem, it is bounded on $L^2(\mathbf{R}^n)$; thus by the converse implication of the $T1$ theorem, it satisfies the assumptions of Figiel's theorem, and so is bounded on $L^p_X(\mathbf{R}^n)$ for $1 < p < \infty$ and all UMD spaces $X$. It seems that this result has not been explicitly stated before, despite the many boundedness results for classical operators which follow from it in a by-now well-known manner, cf. [7]. Among them is the $L^p_X(\mathbf{R})$-boundedness of the Cauchy integral on a Lipschitz graph, $C_A f(x) := \text{p.v.-}\int_{-\infty}^{\infty} (x + \mathrm{i}A(x) - y - \mathrm{i}A(y))^{-1} f(y)\,\mathrm{d}y$, $A' \in L^\infty_{\mathbf{R}}(\mathbf{R})$, which for $X = \mathbf{C}$ is a celebrated theorem of R. Coifman, A. McIntosh and Y. Meyer [6]. The case when $X$ is a UMD lattice was obtained by J.L. Rubio de Francia as a corollary of a general theorem for such spaces [18]. We obtain all these results by a different method, which does not rely on the scalar versions of the theorems, but provides also a new proof of the $Tb$ theorem in the scalar case.

Let us then return to the operator-valued kernels. If $X$ and $Y$ are both Hilbert spaces, the unit-ball of $\mathcal{L}(X, Y)$ is already $R$-bounded, and hence the "$R$-conditions" again reduce to their uniform analogues. (More generally, this remark applies if $X$ has cotype 2 and $Y$ has type 2; cf. [1].) However, we are still left with the requirements that $\tilde{w} \in BMO_U(\mathbf{R}^n)$, $w \in BMO_V(\mathbf{R}^n)$, where $U$ and $V$ should be UMD spaces, thus strict subspaces of $\mathcal{L}(X, Y)$

and $\mathcal{L}(Y', X')$ unless $X$ or $Y$ is finite-dimensional. (Examples of natural UMD subspaces of $\mathcal{L}(X)$, where $X$ is a Hilbert space, are provided by the Schatten–von Neumann ideals, or "non-commutative $L^p$ spaces," $\mathcal{L}^p(X) := \{u \in \mathcal{L}(X)\colon\ \|u\|_{\mathcal{L}^p} := [\operatorname{tr}(u^*u)^{p/2}]^{1/p} < \infty\}$ for $1 < p < \infty$; cf. [18, Proposition 3(v)].)

On the other hand, essentially the best converse implication we can assert is the membership of $\tilde{w}(\cdot)x$ in $BMO_Y(\mathbf{R}^n)$ uniformly in $x \in B_X$, with a similar condition for $w$. These strong-topology conditions are not sufficient, as is shown by counterexamples of F. Nazarov, G. Pisier, S. Treil and A. Volberg [16,17]. Thus there remains a wide gap between the necessary and the sufficient conditions that we know of. All this subtlety is already present in the $T1$ theorem, and so we refer to [15] for a more detailed discussion. It seems likely that the *BMO*-like conditions of Theorem 1.1 are not the last word on operator-valued $Tb$, and that the "right" assumptions for dealing with operator-valued paraproducts are still to be found. On the other hand, Theorem 1.1 yields a "special $Tb$ theorem" for the case $Tb = T'\tilde{b} = 0$ with rather satisfying assumptions.

The organization of the paper is as follows. In Section 2 we establish essential estimates for paraproduct operators in an abstract martingale context. The next two sections are devoted to preparatory results in the more specific setting of function spaces on $\mathbf{R}^n$. Section 3 takes up the construction of the new bases for the $Tb$ theorem, as in [7], and we prove the basic estimates for the analogues of Figiel's $T$-operators by reducing the question to the situation already treated in [10]. The new versions of Figiel's $U$-operators are then treated in Section 4. In the last three sections we finally apply the developed machinery to the $Tb$ theorem. In Section 5 we construct, in the spirit of [11], the decomposition of an operator into the analogues of the elementary transformations of the four types, and provide abstract conditions to guarantee the boundedness of the parts related to the types (1), (2) and (3). The remaining parts of the operator are identified with bounded paraproducts in Section 6, which completes the proof of our abstract $Tb$ Theorem 6.12. In the final Section 7 we show how Theorem 1.1 follows from this abstract version.

## 2. Paraproducts and twisted martingale differences

In this section we estimate paraproducts and related operators in an abstract martingale setting which we first introduce, together with some preliminaries.

We consider a $\sigma$-finite measure space $(S, \mathcal{F}, \mu)$ with a filtration $(\mathcal{F}_k)_{k=-\infty}^{\infty}$ such that $\mathcal{F}$ is generated by $\bigcup_{-\infty}^{\infty} \mathcal{F}_k$. For any $\sigma$-algebra $\mathcal{G}$, we set $\mathcal{G}^+ := \{G \in \mathcal{G}\colon\ 0 < |G| < \infty\}$, where $|G| := \mu(G)$.

We assume for simplicity that our filtration is *regular* (which is always the case in the subsequent applications): each $\mathcal{F}_k$ is generated by a countable number of *atoms* of finite measure, and there exists a positive constant $B > 0$ such that

$$\forall \text{ atoms } A_i \in \mathcal{F}_i,\ i = k, k+1\colon \quad A_{k+1} \subset A_k \quad \Rightarrow \quad |A_k| \leqslant B|A_{k+1}|. \tag{2.1}$$

It is also assumed that $\inf_{A \in \mathcal{F}_k^+} |A| \to \infty$ as $k \to -\infty$. This implies that $E_k f \to 0$ a.e. and in $L^p$ for all $f \in L^p(\mathcal{F})$, $1 \leqslant p < \infty$. The regularity assumption is not essential for the results of this section but simplifies the arguments.

$L^p_X(\mathcal{G})$ denotes the space of (equivalence classes of) strongly $\mathcal{G}$-measurable $X$-valued functions $f$ with $|f(\cdot)|_X \in L^p(\mathcal{G})$. Here $X$ is some Banach space. The norm of the space $L^p_X(\mathcal{G})$ is usually denoted simply by $\|\cdot\|_p$; only occasionally do we resort to the more precise notation $\|\cdot\|_{p,X}$ when a confusion seems otherwise likely. By local integrability conditions we understand integrability on sets of finite measure, e.g., $L^1_{X,\mathrm{loc}}(\mathcal{G})$ consists of those strongly $\mathcal{G}$-measurable $X$-valued functions with $f1_G \in L^1_X(\mathcal{G})$ for every $G \in \mathcal{G}^+$. We will occasionally employ the expectation symbol $E$ for the integral $Ef \equiv \int_S f\,\mathrm{d}\mu$, even though we are not assuming that $S$ should be a probability space.

For the $\sigma$-algebras $\mathcal{F}_k$, the corresponding conditional expectation operators

$$E_k f := E(f|\mathcal{F}_k) := \sum_{A\in\mathcal{F}_k \text{ atom}} 1_A \int_A f\,\mathrm{d}\mu$$

are well defined for $f \in L^1_{X,\mathrm{loc}}(\mathcal{F})$.

We will also need the difference operators $D_k := E_k - E_{k-1}$ and the maximal operators $M_k f := \sup_{j\leqslant k} |E_j f|$ and $M := M_\infty$.

The martingale space $BMO_X(\mathcal{F})$ of bounded mean oscillation is defined in terms of the norms

$$\begin{aligned}\|w\|_{BMO,p} &:= \sup_j \big\| \big(E_j|w - E_{j-1}w|_X^p\big)^{1/p} \big\|_\infty \\ &= \sup_j \sup_{A\in\mathcal{F}_j^+} |A|^{-1/p} \big\| 1_A(b - E_{j-1}b)\big\|_p.\end{aligned}$$

The fundamental inequality of F. John and L. Nirenberg, proved in the generality of arbitrary filtrations and vector-valued functions by C. Herz [13], says that all these norms are equivalent for $1 \leqslant p < \infty$. Let us write $\|f\|_{BMO} := \|f\|_{BMO,1}$.

For later use we point out the simple but useful estimate

$$\|D_j w\|_\infty = \big\|E_j(w - E_{j-1}w)\big\|_\infty \leqslant \big\|E_j|w - E_{j-1}w|_X\big\|_\infty \leqslant \|w\|_{BMO}. \tag{2.2}$$

The Hardy space $H^1_X(\mathcal{F})$ is the atomic space generated by the functions $a$ which satisfy $a = 1_A a$, $E_k a = 0$ and $|A|\|a\|_\infty \leqslant 1$ for some $k \in \mathbf{Z}$ and $A \in \mathcal{F}_k^+$. We shall exploit the well-known interpolation result, stated in the present setting, e.g., by O. Blasco and Q. Xu [2], which guarantees the boundedness from $L^p_X(\mathcal{F})$ to $L^p_Y(\mathcal{F})$ of any linear operator bounded from $H^1_X(\mathcal{F})$ to $L^1_Y(\mathcal{F})$ and from $L^\infty_X(\mathcal{F})$ to $BMO_Y(\mathcal{F})$.

We now leave the general setting and consider a UMD space $X$. Recall that a Banach space $X$ is said to be UMD (i.e., to have the property of *unconditional martingale differences*) if the two-sided estimate

$$\mathcal{U}^{-1}\Big\|\sum D_k f\Big\|_p \leqslant E_\varepsilon\Big\|\sum \varepsilon_k D_k f\Big\|_p \leqslant \mathcal{U}\Big\|\sum D_k f\Big\|_p, \tag{2.3}$$

is satisfied uniformly for all $f \in L^p_X(\mathcal{F})$, and for all measure spaces and filtrations as described above. The $\varepsilon_k$ always designate independent (of each other, as well as of all the other functions considered) random variables satisfying $P(\varepsilon_k = +1) = P(\varepsilon_k = -1) = \frac{1}{2}$, and $E_\varepsilon$ is the expectation on the probability space on which they are defined.

It is well known that as soon as (2.3) holds for the dyadic filtration of [0, 1], it holds for all probability spaces and with the same constant. That it also holds for the spaces of the kind we here consider follows easily from the local nature of the conditional expectation operators in our study. The best constant $\mathcal{U}$ above depends on $p$, but to simplify notation, we do not indicate this dependence explicitly.

Recall that $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $R$-bounded if

$$E_\varepsilon \Big|\sum \varepsilon_j T_j x_j\Big|_Y \leqslant C E_\varepsilon \Big|\sum \varepsilon_j x_j\Big|_X$$

for all finite subsets $(x_j) \subset X$ and $(T_j) \subset \mathcal{T}$; the smallest $C$ is denoted by $\mathcal{R}(\mathcal{T})$. Note that the contraction principle of J.-P. Kahane, $E|\sum \varepsilon_j \lambda_j x_j|_X \leqslant 2 \sup |\lambda_j| \cdot E|\sum \varepsilon_j x_j|_X$, can be rephrased as $\mathcal{R}(\Lambda \cdot \mathrm{id}) \leqslant 2 \sup_{\lambda \in \Lambda} |\lambda|$ for $\Lambda \subset \mathbf{C}$. We refer to [5,20] for other basic properties.

The UMD-condition implies the $R$-boundedness of the conditional expectation operators, as stated in the following proposition. This result is classical for scalar-valued functions, and due to E.M. Stein [19, Theorem 8] in this case; that it holds for general UMD-spaces was observed by Bourgain in [3]. A proof is found in [12, Lemma 34], where it can also be seen that only the right-hand side estimate in (2.3) is actually needed.

**Proposition 2.4.** (Stein [19] (scalar); Bourgain [3] (UMD)) *Let $X$ be a UMD space, $1 < p < \infty$. Then the set $\mathcal{E} = (E_j)_{j=-\infty}^{\infty}$ of conditional expectations, viewed as operators on $L^p_X(\mathcal{F})$, is $R$-bounded with $\mathcal{R}(\mathcal{E}) \leqslant \mathcal{U}$.*

We record the following basic tools which we will often employ without much notice.

**Lemma 2.5.** *For any Banach space $X$, any $p \in [1, \infty[$, and any $f_j \in L^p_X(\mathcal{F})$, we have*

$$K^{-1} E_\varepsilon \Big\|\sum \varepsilon_j f_j\Big\|_p \leqslant \Big\| E_\varepsilon \Big|\sum \varepsilon_j f_j\Big|_X \Big\|_{L^p} \leqslant K E_\varepsilon \Big\|\sum \varepsilon_j f_j\Big\|_p,$$

*where $K = K_p$ is the constant from Kahane's inequality*

$$E\Big[\Big|\sum \varepsilon_j x_j\Big|_X^p\Big] \leqslant K_p \Big[E\Big|\sum \varepsilon_j x_j\Big|_X\Big]^p.$$

**Lemma 2.6.** *For any Banach spaces $X$, $Y$, any $p \in [1, \infty[$, and $f_j \in L^p_X(\mathcal{F})$, $g_j \in L^\infty_{\mathcal{L}(X,Y)}(\mathcal{F})$, we have*

$$E_\varepsilon \Big\|\sum \varepsilon_j g_j f_j\Big\|_{p,Y} \leqslant K^2 \big\|\mathcal{R}\big(\{g_j(\cdot)\}_{j \in \mathbf{Z}}\big)\big\|_\infty E_\varepsilon \Big\|\sum \varepsilon_j f_j\Big\|_{p,X}.$$

*For scalar-valued $g_j$, one has $\|\mathcal{R}(\{g_j\}_j)\|_\infty \leqslant 2 \sup_j \|g_j\|_\infty$.*

Lemma 2.5 is formulated so as to make the proof rather obvious. Lemma 2.6 follows upon two applications of Lemma 2.5, with the definition of $R$-boundedness in between.

Our *paraproduct* is the bilinear operator defined by the formal series

$$P(w, f) := \sum D_{j+1}w \cdot E_j f.$$

For technical reasons, which become clear later, we consider the more general trilinear object

$$\Pi(b, w, f) := \sum D_{j+1}\big(b(w - E_j w)\big) \cdot E_j f,$$

which reduces to $P(w, f)$ when $b \equiv 1$.

Given UMD spaces $X$ and $Y$, and a UMD $R$-space $U \hookrightarrow \mathcal{L}(X, Y)$, we want to give a meaning to $\Pi(b, w, f)$ as an element of $L^p_Y(\mathcal{F})$ whenever $b \in L^\infty(\mathcal{F})$, $w \in BMO_U(\mathcal{F})$ and $f \in L^p_X(\mathcal{F})$, where $1 < p < \infty$. To this end, we first interpret the series as a linear functional acting on the (dense) subspace of $L^{p'}_{X'}(\mathcal{F})$ consisting of those $g$ for which there is an $N$ such that $D_j g = 0$ for $|j| > N$. The action of $P(w, f)$ on such a $g$ is determined by the finite sum over $|j + 1| \leqslant N$. Thus, to give a meaning to $P(w, f)$ as an element of $L^p_X(\mathcal{F})$, it suffices to show the boundedness of the partial sums of the defining series in this space, so we may and do henceforth assume that the summation in the definition of $\Pi$ is over some finite but arbitrary set of $j$. Let us denote $w_j := b(w - E_j w)$.

The following basic estimate is a generalization of one presented in Figiel and Wojtaszczyk [12], but attributed by them to Bourgain.

**Lemma 2.7.** *Let $X$, $Y$ be UMD spaces, $U \hookrightarrow \mathcal{L}(X, Y)$ a UMD $R$-space, $1 < p < \infty$. Then, for $b \in L^\infty(\mathcal{F})$, $w \in L^p_U(\mathcal{F}) \cap BMO_U(\mathcal{F})$ and $f \in L^p_X(\mathcal{F}) \cap L^\infty_X(\mathcal{F})$, the norm $\|\Pi(b, w, f)\|_{p,Y}$ is dominated by*

$$C\|b\|_\infty\big(\|w\|_{p,U}\|f\|_{\infty,X} + \|w\|_{BMO,U}\|f\|_{p,X}\big) + C\big\|P(b, w)\big\|_{p,Y}\|f\|_{\infty,X}.$$

*In particular,*

$$\big\|P(w, f)\big\|_{p,Y} \leqslant C\|w\|_{p,U}\|f\|_{\infty,X} + C\|w\|_{BMO,U}\|f\|_{p,X}.$$

**Proof.**

$$\begin{aligned}
\big\|\Pi(b, w, f)\big\|_{p,Y} &\leqslant \mathcal{U} E_\varepsilon \Big\|\sum \varepsilon_j D_{j+1}w_j \cdot E_j f\Big\|_{p,Y} \\
&\leqslant \mathcal{U} E_\varepsilon \Big\|\sum \varepsilon_j E_{j+1}[D_{j+1}w_j \cdot f]\Big\|_{p,Y} \\
&\quad + \mathcal{U} E_\varepsilon \Big\|\sum \varepsilon_j D_{j+1}w_j \cdot D_{j+1} f\Big\|_{p,Y} \\
&\leqslant \mathcal{U}^2 E_\varepsilon \Big\|\Big[\sum \varepsilon_j D_{j+1}w_j\Big] f\Big\|_{p,Y}
\end{aligned}$$

$$
\begin{aligned}
&+ 2\mathcal{U}K^2 \big\| \mathcal{R}\big(\{D_{j+1}w_j\}_j\big) \big\|_\infty E_\varepsilon \Big\| \sum \varepsilon_j D_{j+1} f \Big\|_{p,X} \\
&\leqslant \mathcal{U}^2 E_\varepsilon \Big\| \sum \varepsilon_j D_{j+1} w_j \Big\|_{p,U} \|f\|_{\infty,X} \\
&+ 2\mathcal{U}^2 K^2 \mathcal{R}(B_U) \sup_j \|D_{j+1}w_j\|_{\infty,U} \|f\|_{p,X}.
\end{aligned}
$$

The supremum norm $\|D_{j+1}w_j\|_{\infty,U}$ in the last term is readily estimated by

$$
2\big\| E_{j+1}\big(b(w - E_j w)\big) \big\|_{\infty,U} \leqslant 2\|b\|_\infty \|w\|_{BMO,U}.
$$

As for the first term, we have $D_{j+1}w_j = D_{j+1}(bw) - D_{j+1}b \cdot E_j w$, and hence

$$
\begin{aligned}
& E_\varepsilon \Big\| \sum \varepsilon_j D_{j+1} w_j \Big\|_{p,U} \\
&\quad \leqslant E_\varepsilon \Big\| \sum \varepsilon_j D_{j+1}(bw) \Big\|_{p,U} + E_\varepsilon \Big\| \sum \varepsilon_j D_{j+1} b \cdot E_j w \Big\|_{p,U} \\
&\quad \leqslant \mathcal{U}\|bw\|_{p,U} + \mathcal{U}\big\|P(b,w)\big\|_{p,U} \leqslant \mathcal{U}\|b\|_\infty \|w\|_{p,U} + \mathcal{U}\big\|P(b,w)\big\|_{p,U}.
\end{aligned}
$$

(Notice that we have here used $\mathcal{U}$ to designate the maximum of the unconditionality constants of martingale difference sequences on $L^p_X(\mathcal{F})$, $L^p_Y(\mathcal{F})$ and on $L^p_U(\mathcal{F})$.) The last claim follows by specialization to $b = 1$ and observing that $P(1,w) = 0$. $\square$

**Lemma 2.8.** *For $b \in L^\infty(\mathcal{F})$, $w \in BMO_U(\mathcal{F})$ and $a$ an atom of $H^1_X(\mathcal{F})$ on $A \in \mathcal{F}_k$, we have*

$$
\big\|\Pi(b,w,a)\big\|_{1,Y} \leqslant C\|b\|_\infty \|w\|_{BMO,U} + C|A|^{-1/2} \big\| P\big(b, 1_A(w - E_k w)\big)\big\|_{2,U};
$$

*in particular $\|P(w,a)\|_{1,Y} \leqslant C\|w\|_{BMO,U}$.*

**Proof.** By assumption, $a = 1_A a$ with $A \in \mathcal{F}_k$, $E_k a = 0$ and $|A|\|a\|_\infty \leqslant 1$. Then

$$
\begin{aligned}
\Pi(b,w,a) &= \sum_{j>k} D_{j+1}w_j \cdot 1_A E_j a = 1_A \sum_{j>k} D_{j+1}\big(b 1_A(w - E_j w)\big) E_j a \\
&= 1_A(\mathrm{id} - E_{k+1}) \Pi\big(b, 1_A(w - E_k w), a\big),
\end{aligned}
$$

where the last equality follows upon observing that

$$
1_A(w - E_j w) = (\mathrm{id} - E_j)\big(1_A(w - E_k w)\big) \quad \text{for } j > k.
$$

Thus we may estimate

$$\begin{aligned}\big\|\Pi(b,w,a)\big\|_{1,Y}&\leqslant 2|A|^{1/2}\big\|\Pi\big(b,1_A(w-E_kw),a\big)\big\|_{2,Y}\\&\leqslant C|A|^{1/2}\big(\|b\|_\infty\big\|1_A(w-E_kw)\big\|_{2,U}\|a\|_{\infty,X}\\&\quad+\|b\|_\infty\big\|1_A(w-E_kw)\big\|_{BMO,U}\|a\|_{2,X}\\&\quad+\big\|P\big(b,1_A(w-E_kw)\big)\big\|_{2,U}\|a\|_{\infty,X}\big).\end{aligned}$$

The assertion follows upon using the size properties of an atom, the definition of the *BMO* norm, and the boundedness of $w\mapsto 1_A(w-E_kw)$ on $BMO_U(\mathcal{F})$, which follows from

$$\begin{gathered}(\mathrm{id}-E_j)\big(1_A(w-E_kw)\big)=1_A(w-E_{j\vee k})\quad\text{and}\\E_{j+1}\big|1_A(w-E_{j\vee k}w)\big|_U\leqslant E_{j+1}E_{(j\vee k)+1}|w-E_{j\vee k}w|_U\leqslant\|w\|_{BMO,U}.\qquad\square\end{gathered}$$

**Lemma 2.9.** *Let $b$ and $w$ be as above and $g\in L^\infty_X(\mathcal{F})$. Then*

$$\begin{aligned}\big\|\Pi(b,w,g)\big\|_{BMO,Y}&\leqslant C\|b\|_\infty\|w\|_{BMO,U}\|g\|_{\infty,X}\\&\quad+C\sup_k\sup_{A\in\mathcal{F}_k^+}|A|^{-1/2}\big\|P\big(b,1_A(w-E_kw)\big)\big\|_{2,U}\|g\|_{\infty,X};\end{aligned}$$

*in particular, $\|P(w,g)\|_{BMO,Y}\leqslant C\|w\|_{BMO,U}\|g\|_{\infty,X}$.*

**Proof.** For $A\in\mathcal{F}_k$, we compute

$$\begin{aligned}1_A(\mathrm{id}-E_{k-1})\Pi(b,w,g)&=1_AD_kw_{k-1}\cdot E_{k-1}g+\sum_{j\geqslant k}D_{j+1}(1_Aw_j)\cdot E_j(1_Ag)\\&=1_AD_kw_{k-1}\cdot E_{k-1}g+(\mathrm{id}-E_k)\Pi\big(b,1_A(w-E_kw),1_Ag\big).\end{aligned}$$

Therefore

$$\begin{aligned}&\big\|1_A(\mathrm{id}-E_{k-1})\Pi(b,w,g)\big\|_{2,Y}\\&\quad\leqslant|A|^{1/2}\|D_kw_{k-1}\|_{\infty,\mathcal{L}(X,Y)}\|g\|_{\infty,X}\\&\qquad+C\|b\|_\infty\big(\big\|1_A(w-E_kw)\big\|_{2,U}\|1_Ag\|_{\infty,X}+\big\|1_A(w-E_kw)\big\|_{BMO,U}\|1_Ag\|_{2,X}\big)\\&\qquad+C\big\|P\big(b,1_A(w-E_kw)\big)\big\|_{2,U}\|g\|_{\infty,X},\end{aligned}$$

from which the asserted estimate readily follows. $\square$

**Proposition 2.10.** *Let $X$ and $Y$ be UMD spaces and $U\hookrightarrow\mathcal{L}(X,Y)$ a UMD $R$-space. Let $b\in L^\infty(\mathcal{F})$ and $w\in BMO_U(\mathcal{F})$. Then*

$$\Pi(b,w,\cdot):\begin{cases}H^1_X(\mathcal{F})\to L^1_Y(\mathcal{F}),\\L^\infty_X(\mathcal{F})\to BMO_Y(\mathcal{F}),\\L^p_X(\mathcal{F})\to L^p_Y(\mathcal{F}),\quad 1<p<\infty,\end{cases}$$

*the norm being bounded by $C\|b\|_\infty\|w\|_{BMO,U}$ in each case, where $C$ depends only on the spaces and the exponent $p$. In particular, these boundedness results hold for the paraproduct $P(w,\cdot)$.*

**Proof.** We have already proved the paraproduct boundedness in the end-point situations, and the estimate

$$\big\|P(w,f)\big\|_{p,Y}\leqslant C\|w\|_{BMO,U}\|f\|_{p,X}$$

follows by interpolation.

We may now apply this result, with $(b,w,\mathbf{C},U,U)$ in place of $(w,f,U,X,Y)$, to the bounds obtained for $\varPi$ in Lemmata 2.8 and 2.9. This readily gives

$$\big\|\varPi(b,w,a)\big\|_{1,Y}\leqslant C\|b\|_\infty\|w\|_{BMO,U},\quad a\text{ an atom of }H^1_X(\mathcal{F}),$$
$$\big\|\varPi(b,w,g)\big\|_{BMO,Y}\leqslant C\|b\|_\infty\|w\|_{BMO,U}\|g\|_{\infty,X}.$$

Another use of interpolation completes the proof. □

As a first application of the paraproduct boundedness, we obtain an unconditionality property of "twisted" martingale differences on UMD spaces. We first define the following notion.

**Definition 1** *($(\mathcal{F}_j)$-accretivity).* A function $b\in L^\infty(\mathcal{F})$ is said to be *$(\mathcal{F}_j)$-accretive* provided that $|E_jb|\geqslant\delta>0$ for all $j$ (thus also $|b|\geqslant\delta$ a.e.).

For such a $b$, we define the *twisted conditional expectations*

$$F_jf:=(E_jb)^{-1}E_j(fb),$$

and the corresponding differences

$$\begin{aligned}\varDelta_jf&:=F_jf-F_{j-1}f\\&=(E_jb)^{-1}D_j(fb)-(E_jb\cdot E_{j-1}b)^{-1}D_jb\cdot E_{j-1}(fb).\end{aligned}\tag{2.11}$$

One verifies by direct computations that the basic properties $F_jF_\ell=F_{j\wedge\ell}$ and $\varDelta_j\varDelta_\ell=\delta_{j\ell}\varDelta_j$ are inherited by the $F_j$ and $\varDelta_j$ from the original operators $E_j$ and $D_j$. The two projections $F_j$ and $E_j$ have the same range, and, moreover,

$$\sum\varDelta_jf=\lim_{m,n\to\infty}\big[(E_nb)^{-1}E_n(bf)-(E_{-m}b)^{-1}E_{-m}(bf)\big]=b^{-1}(bf)-0=f$$

by martingale convergence.

**Proposition 2.12.** *For an $(\mathcal{F}_j)$-accretive $b \in L^\infty(\mathcal{F})$, the related twisted martingale differences are unconditional on $L^p_X(\mathcal{F})$, more precisely*

$$C^{-1} E_\varepsilon \left\| \sum \varepsilon_j \Delta_j f \right\|_p \leqslant \left\| \sum \Delta_j f \right\|_p \leqslant C E_\varepsilon \left\| \sum \varepsilon_j \Delta_j f \right\|_p$$

*where $\Delta_j$ is defined in* (2.11).

**Proof.** Using (2.11) one gets the left inequality from the contraction principle, the UMD-property and the paraproduct boundedness. The right inequality is obtained by a standard duality argument with $g \in L^{p'}_{X'}(\mathcal{F})$. □

As a simple consequence, we obtain the boundedness of martingale transforms with respect to the twisted difference sequences.

**Corollary 2.13.** *Let $v_j \in L^\infty(\mathcal{F}_j)$. Then*

$$\left\| \sum v_{j-1} \Delta_j f \right\|_p \leqslant C \sup_j \|v_j\|_\infty \left\| \sum \Delta_j f \right\|_p.$$

**Proof.** It is easily seen that $v_{j-1}\Delta_j f = \Delta_j(v_{j-1} f)$. Thus Proposition 2.12 and the contraction principle give

$$\begin{aligned} \left\| \sum v_{j-1} \Delta_j f \right\|_p &\leqslant C E_\varepsilon \left\| \sum \varepsilon_j v_{j-1} \Delta_j f \right\|_p \\ &\leqslant 2K^2 C \sup_j \|v_j\|_\infty E_\varepsilon \left\| \sum \varepsilon_j \Delta_j f \right\|_p, \end{aligned}$$

and another application of Proposition 2.12 completes the proof. □

Up to this point, we used the paraproducts as a device to establish the unconditionality property and the boundedness of martingale transforms with respect to the twisted martingale difference sequences. We still need to go one step further to generalize the paraproduct boundedness result to *twisted paraproducts*.

We begin with the definition. With the $(\mathcal{F}_j)$-accretive function $b$ fixed, we consider the operator

$$\mathcal{P}(w, f) = \sum \Delta_{j+1} w \cdot F_j f, \tag{2.14}$$

which is indeed recognized as a twisted analogue of $P(w, f)$ in that all the conditional expectations have been replaced by their twisted versions.

**Proposition 2.15.** *Let $X$, $Y$ be UMD spaces, $U \hookrightarrow \mathcal{L}(X, Y)$ be a UMD $R$-space, and $1 < p < \infty$. Let $b \in L^\infty(\mathcal{F})$ be an $(\mathcal{F}_j)$-accretive function, and $\mathcal{P}$ denote the twisted para-*

*product defined in terms of the b-twisted conditional expectation operators as in* (2.14). *Then for* $w \in BMO_U(\mathcal{F})$ *and* $f \in L^p_X(\mathcal{F})$ *we have*

$$\mathcal{P}(w, f) \in L^p_Y(\mathcal{F}) \quad \textit{and} \quad \big\|\mathcal{P}(w, f)\big\|_{p,Y} \leqslant C\|w\|_{BMO,U}\|f\|_{p,X}.$$

**Proof.** By argumentation similar to that in the case of the original paraproduct, we may and do restrict, in seeking $L^p$ estimates for $\mathcal{P}(w, f)$, to the situation were the summation in (2.14) is over finitely many indices only.

Since $F_j E_j = E_j$ and $\Delta_{j+1} F_j = 0$, we notice that

$$\begin{aligned}
\Delta_{j+1} w &= \Delta_{j+1}(w - E_j w) \\
&= \big[(E_{j+1}b)^{-1} - (E_j b)^{-1}\big] E_{j+1}\big(b(w - E_j w)\big) + (E_j b)^{-1} D_{j+1}\big(b(w - E_j w)\big) \\
&= -(E_{j+1}b \cdot E_j b)^{-1} D_{j+1} b \cdot E_{j+1}\big(b(w - E_j b)\big) \\
&\quad + (E_j b)^{-1} D_{j+1}\big(b(w - E_j w)\big).
\end{aligned}$$

Therefore by unconditionality of Proposition 2.12 and the contraction principle

$$\begin{aligned}
\big\|\mathcal{P}(w, f)\big\|_p &\leqslant C E_\varepsilon \Big\| \sum \varepsilon_j \Delta_{j+1} w \cdot F_j f \Big\|_p \\
&\leqslant C K^2 \delta^{-3} E_\varepsilon \Big\| \sum \varepsilon_j E_{j+1}\big(b(w - E_j w)\big) \cdot D_{j+1} b \cdot E_j(bf) \Big\|_p \\
&\quad + C K^2 \delta^{-2} E_\varepsilon \Big\| \sum \varepsilon_j D_{j+1}\big(b(w - E_j w)\big) \cdot E_j(bf) \Big\|_p.
\end{aligned}$$

In the second term here we have the randomized $L^p$ norm of $\Pi(b, w, bf)$, for which we have the bound $C\|b\|_\infty^2 \|w\|_{BMO,U}\|f\|_{p,X}$.

Consider the quantities $E_{j+1}(b(w - E_j w))$, $j \in \mathbf{Z}$, appearing in the first term. We have

$$\big|E_{j+1}\big(b(w - E_j w)\big)\big|_U \leqslant E_{j+1}\big[\|b\|_\infty |w - E_j w|_U\big] \leqslant \|b\|_\infty \|w\|_{BMO,U}.$$

Since $U$ is an $R$-space, this uniform boundedness in $U$ implies that

$$\mathcal{R}\big(\big\{E_{j+1}\big(b(w - E_j w)\big)\big\}_{j=-\infty}^{\infty}\big) \leqslant \mathcal{R}(B_U)\|b\|_\infty \|w\|_{BMO,U}$$

almost everywhere, and hence

$$\begin{aligned}
&E_\varepsilon \Big\| \sum \varepsilon_j E_{j+1}\big(b(w - E_j w)\big) \cdot D_{j+1} b \cdot E_j(bf) \Big\|_p \\
&\quad \leqslant C\|w\|_{BMO,U} E_\varepsilon \Big\| \sum \varepsilon_j D_{j+1} b \cdot E_j(bf) \Big\|_p.
\end{aligned}$$

What remains here is the randomized $L^p$ norm of the usual paraproduct $P(b, bf)$, for which we already know the estimate

$$C\|b\|_{BMO}\|bf\|_p \leqslant C\|b\|_\infty^2 \|f\|_p.$$

This completes the proof. $\square$

## 3. Bases for functions on the Euclidean space

Recall that $b \in L^\infty(\mathbf{R}^n)$ is called *para-accretive* provided that the following condition holds. There are $C, \delta > 0$ such that for all $x \in \mathbf{R}^n$ and $r > 0$ there is a cube $Q$ with $d(x, Q) \leqslant Cr$, $C^{-1}r \leqslant \operatorname{diam} Q \leqslant Cr$ such that

$$\frac{1}{|Q|}\left|\int_Q b(y)\,\mathrm{d}y\right| \geqslant \delta. \tag{3.1}$$

We shall exploit the fact that a para-accretive $b$ is $(\mathcal{F}_j)$-accretive for an appropriate filtration of the measure space $(\mathbf{R}^n, \mathcal{B}, m)$, where $\mathcal{B}$ is the Borel $\sigma$-algebra and $m$ the Lebesgue measure.

First we record the following result from the context of the "classical" $Tb$-theorem [7].

**Lemma 3.2.** [7] *Under the above assumption, a family $\mathcal{Q} = \bigcup_{-\infty}^{\infty} \mathcal{Q}_j$ of "cubes" exists with the following properties*:

- *Each $\mathcal{Q}_j$ is a partition of $\mathbf{R}^n$.*
- *If $Q \in \mathcal{Q}_j$, then one of the following possibilities holds* (*for the parameter $N$, one can take any natural number larger than some $N_0$*):
  - *$Q$ is a dyadic cube with $\ell(Q) :=$ side-length of $Q = 2^{-2jN}$ or $\ell(Q) = 2^{-(2j+1)N}$, or*
  - *$Q = Q_1 \setminus Q_2$, where $Q_1$, respectively $Q_2$, satisfies the first, respectively second condition above.*
- *For every $Q \in \mathcal{Q}$, condition* (3.1) *holds with a new $\delta' > 0$ in place of $\delta$.*
- *If $Q_1, Q_2 \in \mathcal{Q}$ are not disjoint, then one is contained in the other.*

If $Q \in \mathcal{Q}_j$, we say that $Q$ is of the $j$th generation and write $\operatorname{gen}(Q) = j$. The cubes of the next (i.e., $(j+1)$th) generation which are contained in $Q$ are called the *daughters* of $Q$, and we denote them by $Q_\eta$, where $0 \leqslant \eta < d_Q$ (:= the number of daughters of $Q$). Note that $2^{nN} \leqslant d_Q \leqslant 2^{2nN} =: d$. We choose $N$ large enough to ensure that $d_Q > 2$ for all $Q \in \mathcal{Q}$ (i.e., $N \geqslant 2$ if $n = 1$, and $N \geqslant 1$ if $n > 1$, a rather mild restriction).

It is useful to introduce the following labelling of the cubes $Q \in \mathcal{Q}$ in terms of the family $\mathcal{D}$ of $2^{2N}$-adic cubes. By definition, every $Q$ of the $j$th generation is contained in a unique minimal $D \in \mathcal{D}$, and we denote $\operatorname{dy}(Q) := D$. If $Q$ itself is a dyadic cube, we define $\operatorname{tp}(Q)$ (:= "type" of $Q$) $:= 0$, and else (i.e., $Q$ is a difference of two dyadic cubes), we let $\operatorname{tp}(Q) := 1$. When the set $\mathcal{Q}$ is fixed, it is clear that $\operatorname{dy}(Q)$ and $\operatorname{tp}(Q)$ determine $Q$ uniquely. Conversely, for every $D \in \mathcal{D}$ there exists a $Q \in \mathcal{Q}$ with $\operatorname{dy}(Q) = D$ and $\operatorname{tp}(Q) = 0$; a $Q \in \mathcal{Q}$ with $\operatorname{dy}(Q) = D$ and $\operatorname{tp}(Q) = 1$ may or may not exist. In accordance with this, for $D \in \mathcal{D}$ and $\alpha \in \mathbf{Z}_2$, we denote by $\operatorname{cb}(D, \alpha)$ the unique $Q \in \mathcal{Q}$ with $\operatorname{dy}(Q) = D$ and $\operatorname{tp}(Q) = \alpha$ if it exists, and $\operatorname{cb}(D, \alpha) := \emptyset$, otherwise.

A useful family of functions on each cube $Q \in \mathcal{Q}$ is next shown to exist. These functions share the main properties of those which are similarly denoted in [7]; however, we also require some additional properties and give a different construction. In the following, we denote $\beta_R \equiv \beta(R) := \int_R b\,\mathrm{d}x$.

**Definition 2** *(Associated martingale).* For $Q \in \mathcal{Q}_j$, a sequence of functions $h_Q^\theta$, $0 \leqslant \theta < d_Q$, is called a *martingale associated with* $Q$, provided it verifies the following properties:

- They are all supported on $Q$, and constant on every $Q_\eta$.
- $h_Q^0$ is a non-zero constant on $Q$, while the other $h_Q^\theta$'s take exactly two non-zero values.
- The sets $Q_i^\theta$, $i = 1, 2$, where $h_Q^\theta$, $0 < \theta < d_Q$, attains its non-zero values satisfy $|\beta(Q_i^\theta)| \approx |Q|$, and the non-zero values themselves are comparable to $|Q|^{-1/2}$.
- $\operatorname{supp} h_Q^1 = Q$, and $|\beta(Q_1^1) - \beta(Q_2^1)| \approx |Q|$.
- If $1 < \theta < d_Q$, then there is $1 \leqslant \theta' < \theta$ so that $\operatorname{supp} h_Q^\theta$ coincides with a set on which $h_Q^{\theta'}$ attains one of its non-zero values.
- $(h_Q^\theta)_{0<\theta<d_Q}$ is a $b$-twisted martingale difference sequence with respect to its generated filtration on $(Q, \mathcal{Q}_{j+1}, m)$.
- The constant of accretivity of $b$ with respect to this filtration is at least $\delta_0 = \delta_0(N, \delta') > 0$.
- For the "twisted scalar product" $\langle f, g\rangle_b := \int f \cdot g \cdot b\,\mathrm{d}x$ there is the following "orthonormality" relation: $\langle h_Q^\theta, h_Q^\zeta\rangle_b = \delta_{\theta\zeta}$.

Before going into the proof of existence, we present an algebraic lemma.

**Lemma 3.3.** *Let $y_i$, $i = 1, \ldots, k$, be complex numbers (or vectors of any normed space just as well) with $|y_i| \geqslant 1$ for all $i$, and also $|\sum y_i| \geqslant 1$, where $k \geqslant 3$. Then there are disjoint sets $I, J$ with $I \cup J = \{1, \ldots, k\}$ such that*

$$\Bigl|\sum_I y_i\Bigr|,\ \Bigl|\sum_J y_i\Bigr|,\ \Bigl|\sum_I y_i - \sum_J y_i\Bigr| \geqslant 1/5.$$

**Proof.** Denote $\sigma := \sum_{i=1}^k y_i$. Then the claim is that

$$\Bigl|\sum_I y_i\Bigr|,\ \Bigl|\sum_I y_i - \sigma\Bigr| \geqslant 1/5, \qquad \Bigl|\sum_I y_i - \sigma/2\Bigr| \geqslant 1/10$$

for some $I \subset \{1, \ldots, k\}$.

Consider the following cases first:

(1) Suppose there is a set $T$ of three indices $i$ such that $|\sigma - 2y_i| < 5^{-1}$. Then

$$\Bigl|\sum_T y_i - \alpha\sigma\Bigr| = \Bigl|\sum_T (y_i - \sigma/2) + (3/2 - \alpha)\sigma\Bigr|$$

$$\geqslant -3/10 + (3/2 - \alpha)$$
$$\geqslant -3/10 + 1/2 = 1/5 \quad \text{for } \alpha = 0, 2^{-1}, 1.$$

Thus $I := T$ will do.

(2) Suppose there is a set $S$ of two indices $i$ such that $|\sigma - 2y_i| < 5^{-1}$, and $j \notin S$ such that $|\sigma - y_j| < 5^{-1}$. Then

$$\left|\sum_S y_i + y_j - \alpha\sigma\right| = \left|\sum_S (y_i - \sigma/2) + (y_j - \sigma) + (2-\alpha)\sigma\right|$$
$$\geqslant -2/5 + 2 - \alpha > 1/5.$$

Thus $I := S \cup \{j\}$ will do.

(3) Suppose there is a set $R$ of two indices $i$ such that $|\sigma - y_i| < 5^{-1}$. Then

$$\left|\sum_R y_i - \alpha\sigma\right| = \left|\sum_R (y_i - \sigma) + (2-\alpha)\sigma\right| \geqslant -2/5 + (2-\alpha) > 1/5.$$

So $I := R$ will do.

If condition (1) does not hold, then there is $j \in \{1, 2, 3\}$ such that $|\sigma - 2y_j| \geqslant 5^{-1}$. If also $|\sigma - y_j| \geqslant 5^{-1}$, then $I := \{j\}$ will do. Suppose $|\sigma - y_j| < 5^{-1}$ then. If condition (2) does not hold, then there is $i \in \{1, 2, 3\} \setminus \{j\}$ such that $|\sigma - 2y_i| \geqslant 5^{-1}$. If condition (3) does not hold, then we must also have $|\sigma - y_i| \geqslant 5^{-1}$. So in this case $I := \{i\}$ will do. □

**Lemma 3.4.** *For every $Q \in \mathcal{Q}$, there exists an associated martingale.*

**Proof.** Denote for short $\beta_\eta := \beta_{Q_\eta}$. We apply the previous lemma to the numbers $\beta_\eta$ in place of the $y_i$. Since $|\sum \beta_\eta| = |\beta_Q| \geqslant \delta'|Q|$ and $|\beta_\eta| \geqslant \delta'|Q_\eta| \geqslant \delta' 2^{-2nN}|Q|$, and the number of subcubes is at least three, we conclude the existence of a subset $I \subset \{0, \dots, d_Q - 1\}$ such that $\sum_{\eta \in I} \beta_\eta$, $\sum_{\eta \in I^c} \beta_\eta$ as well as $\sum_{\eta\in I} \beta_\eta - \sum_{\eta \in I^c} \beta_\eta$ all have absolute values not less than $5^{-1} 2^{-2nN} \delta' |Q|$.

We take $h_Q^0 := \beta_Q^{-1/2} 1_Q$ which is the unique (up to sign) choice of $h_Q^0$. Denote $R := \bigcup_{\eta \in I} Q_\eta$, and take $h_Q^1 := y_1 1_R + y_2 1_{Q \setminus R}$, where the $y_i$ are chosen so as to meet the conditions

$$\langle h_Q^1, h_Q^0 \rangle_b = 0 \quad \text{and} \quad \langle h_Q^1, h_Q^1 \rangle_b = 1.$$

This gives $y_1 = \beta_{Q\setminus R}\lambda$, $y_2 = -\beta_R \lambda$, with $\lambda = (\beta_R \beta_{Q\setminus R} \beta_Q)^{-1/2}$. Since all the $\beta$-quantities are comparable to $|Q|$, we obtain the asserted order of magnitude of the non-zero values of $h_Q^1$. We also see that not only is $y_1 \neq y_2$ but in fact $y_1 - y_2 = \lambda(\beta_{Q\setminus R} + \beta_R) = \lambda \beta_Q$ is also comparable to $|Q|^{-1/2}$ in absolute value.

In order to define the rest of the $h_Q^\theta$'s we simply proceed recursively: take one of the maximal sets $J$ such that all the previous $h_Q^\theta$'s are constant on $C = \bigcup_{\eta \in J} Q_\eta$. We can assume by induction that $|\beta_C| \geqslant c|Q|$. Then we apply the same procedure with $C$ and $J$ in place of $Q$ and $\{0, \dots, d_Q - 1\}$ as above to complete the proof. The only difference is that $J$ may not satisfy $\#J \geqslant 3$; however, it is plain that everything except for the last estimate in the previous lemma also holds for $k = 2$, and this last estimate was only used to ensure the condition $|\beta(Q_1^1) - \beta(Q_2^1)| \approx |Q|$, which is not required for $\theta \neq 1$. $\square$

If $f$ is any function (even vector-valued) supported on $Q$ and constant on every $Q_\eta$, and such that $\int f \cdot b \,\mathrm{d}x = 0$, then one obtains the representation

$$f = \sum_{1 \leqslant \theta < d_Q} \langle f, h_Q^\theta \rangle_b h_Q^\theta, \tag{3.5}$$

and the coefficients are uniquely determined by $f$ (cf. [7]).

**Proposition 3.6.** *Let $1 < p < \infty$ and $X$ be a UMD space. For $f \in L_X^p(\mathbf{R}^n)$, we have*

$$f = \sum_{Q \in \mathcal{Q}} \sum_{1 \leqslant \theta < d_Q} \langle f, h_Q^\theta \rangle_b h_Q^\theta \tag{3.7}$$

*with unconditional convergence*; *in fact,*

$$C^{-1} \|f\|_p \leqslant E_\varepsilon \left\| \sum_Q \sum_\theta \varepsilon_Q^\theta \langle f, h_Q^\theta \rangle_b h_Q^\theta \right\|_p \leqslant C \|f\|_p. \tag{3.8}$$

**Proof.** Denote by $\Delta_j$ the twisted martingale difference projections related to the filtration $(\sigma(\mathcal{Q}_j))$, and by $\Delta_Q^\theta f := \langle f, h_Q^\theta \rangle_b h_Q^\theta$ the difference projections of the finer filtration related to the associated martingales as in Definition 2.

It suffices to prove the estimate for all $f = \sum_j \sum_{Q \in \mathcal{Q}_j} 1_Q \Delta_j f$ with a finite double sum, as these constitute a dense subset. Representation (3.5) is valid for every $1_Q \Delta_{j+1} f$, $Q \in \mathcal{Q}_j$. Hence

$$\Delta_{j+1} f = \sum_{Q \in \mathcal{Q}_j} 1_Q \Delta_{j+1} f = \sum_{Q \in \mathcal{Q}_j} \sum_{1 \leqslant \theta < d_Q} \langle 1_Q \Delta_{j+1} f, h_Q^\theta \rangle_b h_Q^\theta.$$

Furthermore,

$$\langle 1_Q \Delta_{j+1} f, h_Q^\theta \rangle_b = \langle \Delta_{j+1} f, h_Q^\theta \rangle_b = \langle F_{j+1} f, h_Q^\theta \rangle_b - \langle F_j f, h_Q^\theta \rangle_b = \langle f, h_Q^\theta \rangle_b - 0.$$

Estimate (3.8) now follows from Proposition 2.12, since $\langle f, h_Q^\theta \rangle_b h_Q^\theta$ are differences of twisted conditional expectations by construction. $\square$

Next, let $\tilde{b} \in L^\infty(\mathbf{R}^n)$ be another para-accretive function. $\tilde{\mathcal{Q}}$ designates the corresponding family of cubes, as in Lemma 3.2. Since the number $N$ appearing in the properties of these cubes can assume any sufficiently large value, we can take same number $N$ for both

$\mathcal{Q}$ and $\tilde{\mathcal{Q}}$. Finally, let $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$, for $\tilde{Q} \in \tilde{\mathcal{Q}}$ and $0 \leqslant \tilde{\theta} < \tilde{d}_{\tilde{Q}}$, denote the basis functions related to the new filtration.

For the later purposes, we require good estimates for the norms of a family of linear operators which map the basis functions $h_Q^\theta$ into $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$ according to certain rules. To be more precise, let $\phi$ be a permutation of $\mathcal{D}$, and $\kappa = (\alpha, \tilde{\alpha}, \vartheta, \tilde{\vartheta}) \in \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_d' \times \mathbf{Z}_d'$, where $\mathbf{Z}_d' := \mathbf{Z}_d \setminus \{0\}$. We are only interested in permutations of the special kind defined as follows:

**Definition 3** *(Rigid permutation).* A permutation $\phi$ of $\mathcal{D}$ is said to be *rigid* if $|\phi(D)| = |D|$ for all $D \in \mathcal{D}$; i.e., every $\phi(D)$ is obtained from $D$ by a rigid motion.

Let then $T_\phi^\kappa$ denote the linear operator on finite linear combinations (with $X$-valued coefficients) of the $h_Q^\theta$ which maps $h_{\mathrm{cb}(D,\alpha)}^{\vartheta}$ to $\tilde{h}_{\tilde{\mathrm{cb}}(\phi(D),\tilde{\alpha})}^{\tilde{\vartheta}}$, whenever the two functions are defined, and all $h_Q^\theta$ not covered by the previous definition to 0.

The proof of the boundedness properties of such mappings will exploit two key principles: Kahane's contractions, and the boundedness of martingale transforms.

We start by recording several simple observations. Below, boundedness always refers to boundedness on the spaces $L_X^p(\mathbf{R}^n)$, $1 < p < \infty$. Moreover, when we speak of a "linear function which takes certain scalar-valued functions $\psi_j$ to certain other scalar-valued functions $\tilde{\psi}_j$," we mean that every finite linear combination *with $X$-coefficients $x_j \in X$* of the form $\sum x_j \psi_j$ is taken to $\sum x_j \tilde{\psi}_j$.

**Remark 3.9.** All projections $\Pi$ defined by choosing one of the alternatives $\Pi h_Q^\theta \in \{0, h_Q^\theta\}$ for every pair $(Q, \theta)$ are uniformly bounded by Proposition 3.6.

**Remark 3.10.** The constant function $1 \in L^\infty(\mathbf{R}^n)$ satisfies the condition (3.1) of para-accretivity with $\delta = 1$ for *every* measurable set $Q \subset \mathbf{R}^n$. In particular, given the filtration of $(\mathbf{R}^n, \mathcal{B}, m)$ induced by the family $\mathcal{Q}$, Lemma 3.4 also applies to yield a family of functions $(\chi_Q^\theta)_{0 \leqslant \theta < d_Q}$ which satisfies the assertions of that lemma with 1 in place of $b$. From the construction given in the proof we even find that these functions can be chosen in such a way that $|\chi_Q^\theta|$ and $|h_Q^\theta|$ are comparable, with constants of comparison independent of $\theta$ and $Q$. From the contraction principle and Proposition 3.6 it then follows that the linear mappings which take $h_Q^\theta$ to $\chi_Q^\theta$ and $\tilde{\chi}_{\tilde{Q}}^{\tilde{\theta}}$ to $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$, respectively, are bounded.

**Lemma 3.11.** *For $D \in \mathcal{D}$, let $\chi_D := \chi_D^1$ if $D \in \mathcal{Q}$, and*

$$\chi_D := \lambda_D\big(\big|\mathrm{cb}(D,0)\big|1_{\mathrm{cb}(D,1)} - \big|\mathrm{cb}(D,1)\big|1_{\mathrm{cb}(D,0)}\big),$$
$$\lambda_D := \big(\big|\mathrm{cb}(D,0)\big|\big|\mathrm{cb}(D,1)\big||D|\big)^{-1/2},$$

*in the opposite case. Then the linear mappings which take $\chi_Q^\theta$ to $\chi_{\mathrm{dy}(Q)}$ whenever $\theta = \vartheta$, $\mathrm{tp}(Q) = \alpha$, and $\chi_Q^\theta$ is defined, and to zero, otherwise, are bounded.*

The $\chi_D$ so defined is, in either case, a function normalized in $L^2$ with

$$\operatorname{supp}\chi_D = D, \qquad \int \chi_D \, \mathrm{d}x = 0,$$

and $\chi_D$ is constant on every proper subcube $Q \subset D$ with $Q \in \mathcal{D}$, and it takes exactly two different non-zero values, both of which (as well as their difference) are of the order $|D|^{-1/2}$.

**Proof.** From the construction of the functions $h_Q^\theta$ in Lemma 3.4, which we also used to build the functions $\chi_Q^\theta$, we find that the support of $\chi_Q^\theta$ with $\theta > 1$ coincides with a set where a certain $\chi_Q^{\theta'}$, with $1 \leqslant \theta' < \theta$, attains one of its non-zero values. Similarly, if $Q \neq \mathrm{dy}(Q)$, then the support of $\chi_Q^1$ coincides with a set where $\chi_{\mathrm{dy}(Q)}$ attains one of its non-zero values.

With this in mind, let $Q_i$, for $i = 0, 1, 2$, be certain disjoint sets, $I_i$ their respective indicators, and $\beta_i = |Q_i|$. Thus the mapping in question is realized as a composition of a bounded number of operations which take functions of the form

$$\chi_1 = \alpha_1(\beta_2 I_1 - \beta_1 I_2) \quad \text{to} \quad \chi_2 = \alpha_2\big((\beta_1 + \beta_2)I_0 - \beta_0(I_1 + I_2)\big), \tag{3.12}$$

where, moreover, all the (positive) quantities $\alpha_i^{-2}$, $\beta_i$ are comparable.

Let us define the martingale differences (on their generated filtration)

$$\begin{aligned} d_1 &:= \frac{\alpha_1\beta_1}{\beta_0 + \beta_1}\big(\beta_2(I_0 + I_1) - (\beta_0 + \beta_1)I_2\big), \\ d_2 &:= \frac{\alpha_1\beta_2}{\beta_0 + \beta_1}(\beta_0 I_1 - \beta_1 I_0), \end{aligned} \tag{3.13}$$

and the constants

$$\mu_0 := \frac{\alpha_2\beta_0}{\alpha_1\beta_1}, \qquad \mu_1 := -\frac{\alpha_2(\beta_0 + \beta_1 + \beta_2)}{\alpha_1\beta_2}. \tag{3.14}$$

Then a computation shows that $\chi_1 = d_1 + d_2$, $\chi_2 = \mu_0 d_1 + \mu_1 d_2$. Thus $\chi_2$ can be viewed as a martingale transform of $\chi_1$ with a bounded transforming sequence. (By what was said about the orders of magnitude of the $\alpha_i$, $\beta_i$, the $\mu_i$ a comparable to 1 in absolute value.)

What is important to us is the fact that this transformation can be performed simultaneously on all the cubes $D \in \mathcal{D}$, as we are about to show. Indeed, let $d_1^D$, $d_2^D$ denote martingale difference like above when the pair $\chi_1 = \chi_1^D$ and $\chi_2 = \chi_2^D$ represents one of the possibilities $\chi_Q^\theta$, $\chi_Q^{\theta'}$ or $\chi_Q^1$, $\chi_D$ (with $\mathrm{dy}(Q) = D$). Let, moreover, $v_i^D := \mu_i^D 1_D$, where $\mu_b^D$ has obvious meaning now. We then define our filtration by setting $\mathcal{F}_{2j} := \sigma(\mathcal{D}_j)$ (where $\mathcal{D}_j$ is the collection of all $D \in \mathcal{D}$ of the $j$th generation), and

$$\mathcal{F}_{2j+1} = \sigma\big(\mathcal{F}_{2j}, \big\{d_1^D \colon D \in \mathcal{D}_j\big\}\big).$$

Finally, we set $v_{2j+i} := \sum_{D \in \mathcal{D}_j} v_i^D$ for $i = 0, 1$.

Now any linear combination (even with $X$-coefficients) of the functions $d_1^Q$, $Q \in \mathcal{D}_j$, is a martingale difference adapted to $\mathcal{F}_{2j+1}$, while any linear combination of the $d_2^Q$, $Q$ in the same set, is a martingale difference adapted to $\mathcal{F}_{2(j+1)}$.

Thus, the transformation

$$\sum_j \Biggl( \sum_{D\in\mathcal{D}_j} x_D d_1^D + \sum_{D\in\mathcal{D}_j} x_D d_2^D \Biggr) \mapsto \sum_j \Biggl( v_{2j} \sum_{D\in\mathcal{D}_j} x_D d_1^D + v_{2j+1} \sum_{D\in\mathcal{D}_j} x_D d_2^D \Biggr),$$

i.e.,

$$\sum_{D\in\mathcal{D}} x_D \chi_1^D \mapsto \sum_{D\in\mathcal{D}} x_D \bigl( v_0^D d_1^D + v_1^D d_2^D \bigr) = \sum_{D\in\mathcal{D}} x_D \chi_2^D,$$

is produced by a martingale transform with a bounded transforming sequence. Such operators are bounded on $L_X^p(\mathbf{R}^n)$ by the UMD property of $X$. As noted, a composition of a bounded number of these martingale transforms (and a projection of the kind considered in Remark 3.9) produces the mapping whose boundedness we wanted to show. □

Let $\hat{\chi}_D$ denote one of $n$-dimensional Haar functions associated with $D$, e.g., if $D = [a,b[\times D'$, $D' \subset \mathbf{R}^{n-1}$, we may take

$$\hat{\chi}_D := |D|^{-1/2}\bigl(1_{[a,(a+b)/2[} - 1_{[(a+b)/2,b[}\bigr) \otimes 1_{D'}.$$

The linear mapping which takes $\chi_D$ to $\hat{\chi}_D$ (and also its inverse) is bounded. Indeed, the non-zero values of these functions are comparable and their supports agree, so this follows from the contraction principle and the estimate

$$\Bigl\| \sum x_D \tilde{\chi}_D \Bigr\|_p \approx E_\varepsilon \Bigl\| \sum \varepsilon_D x_D \tilde{\chi}_D \Bigr\|_p, \tag{3.15}$$

which holds both for $\tilde{\chi}_D = \chi_D$ and for $\tilde{\chi}_D = \hat{\chi}_D$, since both of these constitute martingale difference sequences.

Moreover, the linear mapping which takes $\hat{\chi}_D$ to $\tilde{\chi}^{\tilde{\vartheta}}_{\check{c}b(D,\tilde{\alpha})}$ if the latter function is defined, and to zero, otherwise, is also bounded for essentially the same reason, as the support of the latter function is contained in $D = \operatorname{supp} \hat{\chi}_D$, and again the non-zero values of both functions are comparable. (Estimate (3.15) for $\tilde{\chi}^{\tilde{\vartheta}}_{\check{c}b(D,\tilde{\alpha})}$ is contained in Proposition 3.6.)

Let us now denote by $\hat{T}_\phi$ the linear operator which maps $\hat{\chi}_D$ to $\hat{\chi}_{\phi(D)}$ for $D \in \mathcal{D}$ (and, e.g., all other $n$-dimensional Haar functions to zero). Combining all the observations made above, we have shown the existence of bounded operators $R_\phi^\kappa$ and $S_\phi^\kappa$, say, with norm bounds independent of $\phi$ and $\kappa$, such that $T_\phi^\kappa = S_\phi^\kappa \circ \hat{T}_\phi \circ R_\phi^\kappa$. Thus the question of boundedness of $T_\phi^\kappa$ is reduced to that of $\hat{T}_\phi$, but this has already been handled by Figiel [10].

In the following, for a dyadic cube $D$, $D^{(k)}$, $k \geqslant 0$, denotes the unique dyadic cube which contains $D$ and is of the generation $\operatorname{gen}(D^{(k)}) = \operatorname{gen}(D) - k$.

**Proposition 3.16.** (Figiel [10]) *Let $\phi$ be a rigid permutation of $\mathcal{D}$ with the following property: there is a $k \in \mathbf{N}$, such that*

$$\text{the closures of } D^{(k)} \text{ and } \phi(D)^{(k)} \text{ intersect for every } D \in \mathcal{D}. \tag{3.17}$$

*Then $\hat{T}_\phi$ is bounded on $L^p_X(\mathbf{R}^n)$ for $X$ a UMD-space and $1 < p < \infty$, and more precisely*

$$\|\hat{T}_\phi\|_{\mathcal{L}(L^p_X(\mathbf{R}^n))} \leqslant C(2+k)^{1/r-1/q}$$

*provided that $L^p_X(\mathbf{R}^n)$ has type $r$ and cotype $q$; for a UMD-space $X$, this is always true with some $1 < r \leqslant 2 \leqslant q < \infty$.*

**Corollary 3.18.** *Under the assumptions of Proposition* 3.16*, the same conclusion also holds with $\hat{T}_\phi$ replaced by $T^\kappa_\phi$.*

## 4. Permutations of the basis functions

Besides $T^\kappa_\phi$, another fundamental operator whose boundedness properties we will exploit is $U^\kappa_\phi$, where $\kappa = (\alpha, \tilde{\alpha}, \vartheta, 0) \in \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}'_d \times \{0\}$ which maps

$$h^\vartheta_{\mathrm{cb}(D,\alpha)} \text{ (if defined)} \mapsto \tilde{h}^0_{\tilde{\mathrm{c}}\mathrm{b}(\phi(D),\tilde{\alpha})} - \left(\frac{\tilde{\beta}_{\tilde{\mathrm{c}}\mathrm{b}(\phi(D),\tilde{\alpha})}}{\tilde{\beta}_{\tilde{\mathrm{c}}\mathrm{b}(D,0)}}\right)^{1/2} \tilde{h}^0_{\tilde{\mathrm{c}}\mathrm{b}(D,0)} \text{ (if defined)},$$

where $\tilde{\beta}_R := \int_R \tilde{b}\,\mathrm{d}x$. As before, all the $h^\theta_Q$'s not covered by this definition are mapped into zero. Note that whether or not the image function above is defined depends solely on the existence of a cube $\tilde{Q} \in \tilde{\mathcal{Q}}$ such that $\mathrm{dy}(\tilde{Q}) = \phi(D)$ and $\mathrm{tp}(\tilde{Q}) = \tilde{\alpha}$.

As above, we start by reducing the problem slightly. From above we already know that the mapping which takes $h^\vartheta_{\mathrm{cb}(D,\alpha)}$, when defined, (to $\chi_D$ and then) to $\tilde{h}^1_{\tilde{\mathrm{c}}\mathrm{b}(D,0)}$ is bounded. Thus the problem is reduced to studying the boundedness of

$$\tilde{h}^1_{\tilde{\mathrm{c}}\mathrm{b}(D,0)} \mapsto \tilde{h}^0_{\tilde{\mathrm{c}}\mathrm{b}(\phi(D),\tilde{\alpha})} - \left(\tilde{\beta}_{\tilde{\mathrm{c}}\mathrm{b}(\phi(D),\tilde{\alpha})}/\tilde{\beta}_{\tilde{\mathrm{c}}\mathrm{b}(D,0)}\right)^{1/2} \tilde{h}^0_{\tilde{\mathrm{c}}\mathrm{b}(D,0)}.$$

Since the properties of the $h$ and $\tilde{h}$-bases are exactly the same, we might just as well drop out the $\tilde{}$'s from our notation and study the operator, which we now simply denote by $U^\alpha_\phi$, taking

$$h^1_{\mathrm{cb}(D,0)} \mapsto h^0_{\mathrm{cb}(\phi(D),\alpha)} - (\beta_{\mathrm{cb}(\phi(D),\alpha)}/\beta_{\mathrm{cb}(D,0)})^{1/2} h^0_{\mathrm{cb}(D,0)}.$$

**Remark 4.1.** Exploiting the boundedness of the linear mapping taking $h^\vartheta_{\mathrm{cb}(D,\alpha)}$ to $\tilde{h}^1_{\tilde{\mathrm{c}}\mathrm{b}(D,0)}$, and the one taking $\tilde{h}^1_{\tilde{\mathrm{c}}\mathrm{b}(C,\tilde{\alpha})}$ to $\tilde{h}^{\tilde{\vartheta}}_{\tilde{\mathrm{c}}\mathrm{b}(C,\tilde{\alpha})}$, we can also reduce the question of boundedness of $T^\kappa_\phi$ (which takes $h^\vartheta_{\mathrm{cb}(D,\alpha)}$ to $\tilde{h}^{\tilde{\vartheta}}_{\tilde{\mathrm{c}}\mathrm{b}(\phi(D),\tilde{\alpha})}$) to that of the mapping $\tilde{h}^1_{\tilde{\mathrm{c}}\mathrm{b}(D,0)} \mapsto \tilde{h}^1_{\tilde{\mathrm{c}}\mathrm{b}(\phi(D),\tilde{\alpha})}$.

Then, by the symmetry of $b$ and $\tilde{b}$, we can just as well study the boundedness of $h^1_{\mathrm{cb}(D,0)} \mapsto h^1_{\mathrm{cb}(\phi(D),\alpha)}$. Let us denote this last operator by $T^{\alpha}_{\phi}$.

In proving the boundedness of the mappings $T^{\kappa}_{\phi}$, we could reduce the matters to a situation, where the result of Figiel stated as Proposition 3.16 was applicable. However, for the proof of the analogous results for the $U^{\kappa}_{\phi}$ introduced above (which we reduced to proving them for the $U^{\alpha}_{\phi}$), it seems unavoidable to repeat and adapt parts of Figiel's proof of Proposition 3.16. As a by-product of this procedure, we also obtain a boundedness proof for the operators $T^{\alpha}_{\phi}$, and thus by Remark 4.1, another proof of Corollary 3.18.

So let $\mathcal{D}$ be the collection of all $2^{2N}$-adic cubes, as in the previous section. We will have to look more closely at the properties of rigid permutations of $\mathcal{D}$ with property (3.17) for some $k \in \mathbf{N}$. When $\phi$ is a rigid permutation of $\mathcal{D}$, the following property will be of interest.

**Definition 4** *($\phi$-compatibility).* Two cubes $C, D \in \mathcal{D}$ are called *$\phi$-compatible*, if the following conditions are satisfied:

- If $|C| = |D|$, then $C$, $D$, $\phi(C)$ and $\phi(D)$ are all disjoint.
- If the cubes have different size, say $|C| > |D|$, and if $D \cup \phi(D)$ intersects with $E$ for $E = C$ or $E = \phi(C)$, then in fact $D \cup \phi(D)$ is contained in one of the grand-daughters of $E$.

A collection $\mathcal{C} \subset \mathcal{D}$ of cubes is $\phi$-compatible if every two cubes $C, D \in \mathcal{C}$ are $\phi$-compatible. (In particular, a single cube $D \in \mathcal{D}$ is $\phi$-compatible if it is $\phi$-compatible with itself, i.e., if $\phi(D) \neq D$.)

**Remark 4.2.** If $C$ and $D$ are $\phi$-compatible, and $|C| \geqslant |D|$, then each of the functions $h^{\theta}_{Q}$ and $\tilde{h}^{\tilde{\theta}}_{\tilde{Q}}$ is a constant on $D \cup \phi(D)$ for $\mathrm{dy}(Q) = C$, $\mathrm{dy}(\tilde{Q}) = \phi(C)$. Indeed, $h^{\theta}_{Q}$ is a constant on the daughters of $Q$ in $\mathcal{Q}$, and these are unions of certain grand-daughters of $\mathrm{dy}(Q)$; a similar remark applies to $\tilde{h}^{\tilde{\theta}}_{\tilde{Q}}$.

The idea of Figiel to prove the boundedness of the mappings $\hat{T}_{\phi}$ and the like, was to look for a partition of $\mathcal{D}$ into subsets which are $\phi$-compatible. We first want to get rid of the set $\mathcal{D}^{=}_{\phi} := \{D \in \mathcal{D}\colon\ \phi(D) = D\}$ of fixed points of $\phi$, whose elements are not $\phi$-compatible even with themselves. We also denote $\mathcal{D}^{\neq}_{\phi} := \mathcal{D} \setminus \mathcal{D}^{=}_{\phi}$, and by $\Pi^{=}_{\phi}$ and $\Pi^{\neq}_{\phi}$ the related projections given by

$$\Pi^{=}_{\phi} : h^{\theta}_{Q} \mapsto 1_{\mathcal{D}^{=}_{\phi}}\big(\mathrm{dy}(Q)\big)h^{\theta}_{Q}, \qquad \Pi^{\neq}_{\phi} = \mathrm{id} - \Pi^{=}_{\phi}.$$

The boundedness of both $T^{\alpha}_{\phi}\Pi^{=}_{\phi}$ and $U^{\alpha}_{\phi}\Pi^{=}_{\phi}$ follows easily: the case of $\alpha = 0$ is trivial, whereas that of $\alpha = 1$ follows readily from Lemma 3.11 and the use of the contraction principle with Proposition 3.6.

Thus we can concentrate our efforts on the set $\mathcal{D}^{\neq}_{\phi}$. Plainly the restriction of $\phi$ to this set is a rigid permutation with no fixed points. In particular, any $D \in \mathcal{D}^{\neq}_{\phi}$ is $\phi$-compatible with itself. We still need to split the set $\mathcal{D}_\phi^{\neq}$, and to this end we follow Figiel by introducing the function

$$a : \mathcal{D} \to \mathbf{Z}_{k+2},\ D \mapsto \operatorname{gen}(D) \mod (k+2).$$

**Remark 4.3.** If $\phi$ is a rigid permutation of $\mathcal{D}$, and $D \in \mathcal{D}_\phi^{\neq}$ satisfies $\phi(D) \subset D^{(k)}$, then any $C \in \mathcal{D}_\phi^{\neq}$ with $a(C) = a(D)$ and $|C| > |D|$ is $\phi$-compatible with $D$.

This follows from the fact that $D \cup \phi(D) \subset D^{(k)}$, which is either a subset of a grand-daughter of, or disjoint with any $E \in \mathcal{D}$ with $|E| > |D|$ and $a(E) = a(D)$.

Note that the function $a$, which induces a partition of $\mathcal{D}_\phi^{\neq}$, only depends on the parameter $k \in \mathbf{N}$, so this one function induces a partition of $\mathcal{D}_\phi^{\neq}$ with the above compatibility property simultaneously for all permutations $\phi$ of the kind described.

Another useful function is $b = b_\phi : \mathcal{D} \to \mathbf{Z}_3$, which is chosen in dependence on the permutation $\phi$ of interest, and defined as follows. On every orbit of $\phi$ with an even number of elements, we alternate the values 0 and 1. On every orbit of $\phi$ with an odd number of elements, we use the value 2 once, and then alternate 0 and 1. (An infinite orbit could be regarded as either one of these.)

The main purpose of $b$ is to handle the compatibility of cubes of the same generation:

**Remark 4.4.** If $\phi$ is a rigid permutation of $\mathcal{D}$, then any $C, D \in \mathcal{D}_\phi^{\neq}$ for which $|C| = |D|$ and $b_\phi(C) = b_\phi(D)$, are $\phi$-compatible.

As for cubes of different generations, the things are slightly more subtle. Let us denote

$$\mathcal{D}^0 := \{D \in \mathcal{D}\colon\ \bar{D} \subset \operatorname{int} D^{(k)}\}, \qquad \mathcal{D}^1 := \mathcal{D} \setminus \mathcal{D}^0,$$
$$\mathcal{D}_\phi^{\epsilon\eta} := \mathcal{D}^\epsilon \cap \phi^{-1}(\mathcal{D}^\eta) \cap \mathcal{D}_\phi^{\neq}$$

for $\epsilon, \eta \in \{0, 1\}$. (Of course, $\bar{R}$ denotes the closure and $\operatorname{int} R$ the interior of the set $R$.)

**Lemma 4.5.** *For $D \in \mathcal{D}_\phi^{00}$, there can be at most two $C \in \mathcal{D}^0$ such that $|C| > |D|$, $a(C) = a(D)$, and $D \cup \phi(D)$ intersects $C$ but is not contained in any grand-daughter of $C$.*

Let us denote by $\mathcal{I}(D)$ the set of such $C$'s. Thus the lemma says that $\#\mathcal{I}(D) \leqslant 2$ for any $D \in \mathcal{D}_\phi^{00}$.

**Proof.** Let $C$ satisfy the properties in the assertion. From $|C| > |D|$ and $a(C) = a(D)$ it follows that $C$ precedes $D$ by at least $k+2$ generations. Thus, if $E \in \{D, \phi(D)\}$ intersects $C$, then it is a subset of a grand-daughter of $C$.

However, by the assumption, $D$ and $\phi(D)$ cannot be subsets of the same grand-daughter of $C$; hence they are either subset of different grand-daughters, or else one is a subset of $C$, while the other is disjoint from $C$. For definiteness, let $E := D$, if this is a subset of $C$, and hence of a grand-daughter $C'$ of $C$, and $E := \phi(D)$, otherwise.

We define the *order* $\operatorname{ord}(Q)$ of $Q \in \mathcal{D}^0$ as the smallest $j \in \mathbf{Z}$ for which $\bar{Q}$ intersects with $\partial R$ for some $R \in \mathcal{D}$ of the $j$th generation. Then obviously $\operatorname{ord}(Q) \leqslant \operatorname{gen}(Q)$, but also $\operatorname{ord}(Q) > \operatorname{gen}(Q) - k$ for $Q \in \mathcal{D}^0$.

Now the closure of $E^{(k)}$ must intersect with $\partial C'$, and hence $\operatorname{ord}(E^{(k)}) \leqslant \operatorname{gen}(C') = \operatorname{gen}(C) + 2$. On the other hand, $\overline{E^{(k)}} \subset \bar{C} \subset \operatorname{int} C^{(k)}$ (where the last inclusion is due to $C \in \mathcal{D}^0$), and hence $\operatorname{ord}(E^{(k)}) > \operatorname{gen}(C^{(k)}) = \operatorname{gen}(C) - k$. It follows that $\operatorname{gen}(C) \in \{\operatorname{ord}(E^{(k)}) - 2, \dots, \operatorname{ord}(E^{(k)}) + k - 1\}$. There is exactly one value in this set which is $\equiv a(C) \bmod (k+2)$, and hence there is exactly one possible generation for $C$.

But in this given generation, there is exactly one cube $C_1$ which contains $D$, and one which contains $\phi(D)$. Thus $C$ is necessarily one of these two cubes $C_1$ and $C_2$. (It might well happen that $C_1 = C_2$.) $\square$

**Corollary 4.6.** *For any subcollection $\mathcal{C}$ of $\mathcal{D}_\phi^{00}$ with the property that every $D \in \mathcal{C}$ is contained in a maximal $C \in \mathcal{C}$, one can define a function $c : \mathcal{C} \to \mathbf{Z}_3$ such that if $C, D \in \mathcal{C}$ and $(a, b, c)(C) = (a, b, c)(D)$, then $C$ and $D$ are $\phi$-compatible. In particular, this holds for every finite subcollection $\mathcal{C} \subset \mathcal{D}_\phi^{00}$.*

It seems that one is forced to pass to a subcollection of the kind described in order to define the function $c$ with the desired properties. In the original proof of Figiel [10] this was avoided, since he dealt with the corresponding operators on $[0, 1]$ instead of $\mathbf{R}^n$, in which case any dyadic interval is contained in a maximal dyadic interval.

**Proof.** Starting from the maximal cubes and proceeding to ever smaller ones, we define recursively

$$c(D) := \min\bigl[\{0, 1, 2\} \setminus \bigl\{c(C)\colon\ C \in \mathcal{C},\ b(C) = b(D) \text{ and } C \in \mathcal{I}(D) \cup \phi^{-1}\bigl(\mathcal{I}(D)\bigr)\bigr\}\bigr].$$

We first note that the set subtracted from $\{0, 1, 2\}$ contains at most two elements. Indeed, $\mathcal{I}(D)$ contains at most two elements $C_1$ and $C_2$, so $C$ must be one of $C_i$ or $\phi^{-1}(C_i)$, $i = 1, 2$. But only one of $C_i$ and $\phi^{-1}(C_i)$ can have the same $b$-value as $D$.

If $|C| = |D|$, then $b(C) = b(D)$ alone ensures the $\phi$-compatibility of $C$ and $D$ by Remark 4.4. If $|C| > |D|$ and $(a, b)(C) = (a, b)(D)$, then by the construction $c(C) \neq c(D)$ if $C \in \mathcal{I}(D)$ or $\phi(C) \in \mathcal{I}(D)$. Thus, if even $c(C) = c(D)$, then neither $C$ nor $\phi(C)$ can intersect with $D \cup \phi(D)$ without containing the whole of this set in one of its grand-daughters. Therefore $C$ is compatible with $D$. $\square$

**Remark 4.7.** We introduce the (obviously commuting) projections

$$\Pi_\phi^{\epsilon\eta} : h_Q^\theta \mapsto 1_{\mathcal{D}_\phi^{\epsilon\eta}}\bigl(\operatorname{dy}(Q)\bigr) h_Q^\theta, \qquad \Pi_a^i : h_Q^\theta \mapsto 1_{\{i\}}\bigl(a \circ \operatorname{dy}(Q)\bigr) h_Q^\theta.$$

It follows readily from Proposition 3.6 that these are bounded operators, with norm bounds independent of $\phi$ and $k$ (which defines the function $a$).

**Lemma 4.8.** *Let $\phi$ be a rigid permutation of $\mathcal{D}$ which satisfies* (3.17). *Then the operators $T_\phi^\alpha \Pi_\phi^{00} \Pi_a^i$ are bounded, with norm bounds uniform in $\phi$ and $k$.*

**Proof.** It suffices to show a uniform bound for operators taking

$$\sum_{D\in\mathcal{C}} x_D h^1_{\mathrm{cb}(D,0)} \to \sum_{D\in\mathcal{C}} x_D h^1_{\mathrm{cb}(\phi(D),\alpha)},$$

where $\mathcal{C} \subset \mathcal{D}_\phi^{00}$ is a momentarily fixed finite subset with the properties that $a(D) = i$ for all $D \in \mathcal{C}$, and, moreover, there exists $\mathrm{cb}(\phi(D), \alpha)$ for all $D \in \mathcal{C}$ (where the last condition is a void constraint if $\alpha = 0$).

Let us consider the mapping $h^1_{\mathrm{cb}(D,0)} \mapsto h^1_{\mathrm{cb}(\phi(D),\alpha)}$ for a single $D \in \mathcal{D}$. The functions $h_1 := h^1_{\mathrm{cb}(D,0)}$ and $h_2 := h^1_{\mathrm{cb}(\phi(D),\alpha)}$ are of the form

$$h_1 = \alpha_1(\beta_2 I_1 - \beta_1 I_2), \qquad h_2 = \alpha_2(\beta_4 I_3 - \beta_3 I_4),$$

where the $I_i$ are indicators of certain disjoint sets $Q_i$ of comparable size, and the complex quantities $\alpha_i$ and $\beta_i := \int_{Q_i} b\,\mathrm{d}x$ are non-zero and satisfy

$$|\alpha_i|^{-2} \approx |\beta_j| \approx |Q_j| \approx |\beta_1 - \beta_2| \approx |\beta_3 - \beta_4|, \tag{4.9}$$

for all $i = 1, 2$ and $j = 1, 2, 3, 4$.

We first note that at least one of the following holds: $\beta_1 + \beta_3 \neq 0 \neq \beta_2 + \beta_4$ or $\beta_1 + \beta_4 \neq 0 \neq \beta_2 + \beta_3$. To see this, assume $\beta_1 + \beta_3 = 0$. Then $\beta_1 + \beta_4 = -\beta_3 + \beta_4 \neq 0$ and $\beta_2 + \beta_3 = \beta_2 - \beta_1 \neq 0$, so second condition holds. The same conclusion follows similarly if $\beta_2 + \beta_4 = 0$. Thus by relabelling the indices if necessary, we may assume that $\beta_1 + \beta_3 \neq 0 \neq \beta_2 + \beta_4$. (In fact, a slightly more careful but similar argument even shows that we could assume $|\beta_1 + \beta_3| \approx |\beta_2 + \beta_4| \approx |\beta_j|$, but this we do not need.)

Now consider the $b$-twisted martingale differences

$$d_1 := \alpha_1\beta_1\beta_2\left(\frac{I_1 + I_3}{\beta_1 + \beta_3} - \frac{I_2 + I_4}{\beta_2 + \beta_4}\right),$$
$$d_2 := \frac{\alpha_1\beta_2}{\beta_1 + \beta_3}(\beta_3 I_1 - \beta_1 I_3) - \frac{\alpha_1\beta_1}{\beta_2 + \beta_4}(\beta_4 I_2 - \beta_2 I_4)$$

(with respect to their generated filtration), and the adapted process given by

$$v_0 := \frac{\alpha_2\beta_3\beta_4}{\alpha_1\beta_1\beta_2}(I_1 + I_2 + I_3 + I_4),$$
$$v_1 := -\frac{\alpha_2\beta_4}{\alpha_1\beta_2}(I_1 + I_3) - \frac{\alpha_2\beta_3}{\alpha_1\beta_1}(I_2 + I_4).$$

Then $h_1 = d_1 + d_2$ and $h_2 = v_0 d_1 + v_1 d_2$, so that $h_2$ is obtained as a martingale transform of $h$. By (4.9), the transforming sequence is bounded by an absolute constant.

Now, just as we did in the proof of Lemma 3.11, we want to do such a martingale transform simultaneously for several cubes $D \in \mathcal{C}$. At this point we use more projections of the type mentioned in Remark 3.9, similar to the $\Pi_a^i$ in Remark 4.7, but now with the functions $b$ and $c$ in place of $a$. Recall from Corollary 4.6 that it is possible to define $\mathbf{Z}_3$-valued functions $b$ and $c$ on $\mathcal{C}$ such that $(b,c)(D) = (b,c)(C)$ for $C, D \in \mathcal{C}$ implies the $\phi$-compatibility of $C$ and $D$. Thus, we can split $\mathcal{C}$ into nine subsets according to the different values of $(b,c)(D) \in \mathbf{Z}_3^2$. Since the corresponding projections $\Pi_b^{i_1} \Pi_c^{i_2}$ are uniformly bounded, it suffices to establish the claim for these subcollections. In other words, we may assume that $\mathcal{C}$ is $\phi$-compatible. But under this assumption the rest of the proof is just like the proof of Lemma 3.11, except that we now have to apply the boundedness of martingale transforms with respect to a *twisted* martingale difference sequence, i.e., Corollary 2.13, instead of the usual martingale transform property which was sufficient in Lemma 3.11. □

Then we do the same with $U_\phi^\alpha$ in place of $T_\phi^\alpha$.

**Lemma 4.10.** *Let $\phi$ be a rigid permutation of $\mathcal{D}$ which satisfies* (3.17). *Then the operators $U_\phi^\alpha \Pi_\phi^{00} \Pi_a^i$ are bounded, with norm bounds uniform in $\phi$ and $k$.*

**Proof.** Now we have to prove a uniform bound for the operators

$$\sum_{D \in \mathcal{C}} x_D h^1_{\mathrm{cb}(D,0)} \mapsto \sum_{D \in \mathcal{C}} x_D \big[ h^0_{\mathrm{cb}(\phi(D),\alpha)} - (\beta_{\mathrm{cb}(\phi(D),\alpha)} / \beta_{\mathrm{cb}(D,0)})^{1/2} h^0_{\mathrm{cb}(D,0)} \big]$$

for every fixed finite collection $\mathcal{C}$, which may be assumed to be $\phi$-compatible and such that the right-hand side makes sense, i.e., the $\mathrm{cb}(\phi(D),\alpha)$'s are defined.

But the argument which shows this is almost the same as the proof of Lemma 3.11. In fact, $h^1_{\mathrm{cb}(D,0)}$ and $U_\phi^\alpha h^1_{\mathrm{cb}(D,0)}$ are of the same form as the $\chi_1$ and $\chi_2$ (respectively) in (3.12), where now $Q_0 = \mathrm{cb}(\phi(D),\alpha)$ and $Q_1 \cup Q_2 = \mathrm{cb}(D,0)$; moreover, $\beta_i = \beta_{Q_i} = \int_{Q_i} b(x)\,\mathrm{d}x$ (instead of $|Q_i|$ used in Lemma 3.11 where, in effect, $b \equiv 1$). Then plainly the same expression (3.13) and (3.14) define martingale differences $d_1, d_2$ and constants $\mu_1$, $\mu_2$ such that $h^1_{\mathrm{cb}(D,0)} = d_1 + d_2$ and $U_\phi^\alpha h^1_{\mathrm{cb}(D,0)} = \mu_0 d_1 + \mu_1 d_2$. There is one subtle point, though, that we must notice. For the defining formulae of $d_1$ and $d_2$ to make sense, we need that $\beta_0 + \beta_1 \neq 0$. (In Lemma 3.11 this was trivial, since both $\beta_0$ and $\beta_1$ were positive numbers.) Here we notice that

$$\big|(\beta_0 + \beta_1) - (\beta_0 + \beta_2)\big| = |\beta_1 - \beta_2| \approx \big|\mathrm{cb}(D,0)\big|$$

by the special property of the pre-images of the two non-zero values of $h^1_{\mathrm{cb}(D,0)}$ stated in Definition 2. From this it follows that at most one of $\beta_0 + \beta_i$, $i = 1, 2$, can vanish; by the symmetry of $Q_1$ and $Q_2$ in the argument we may suppose that $\beta_0 + \beta_1 \neq 0$ by re-indexing, if necessary.

After this observation, the proof copies that of Lemma 3.11. Again, we need the boundedness result for twisted martingale transforms (Corollary 2.13). □

Now we are left with the other projected operators $T_\phi^\alpha \Pi_\phi^{\epsilon\eta}$ and $U_\phi^\alpha \Pi_\phi^{\epsilon\eta}$. We will see that these can be reduced to the case $\epsilon = \eta = 0$ with the help of a simple auxiliary permutation.

Define a permutation $\psi$ of $\mathcal{D}$ with $\psi^2 = \mathrm{id}$ as follows. For every $C \in \mathcal{D}$, permute the cubes in $C^{(-k)} := \{D \in \mathcal{D}\colon\ D^{(k)} = C\}$ in such a way that for $D \in C^{(-k)} \cap \mathcal{D}^1$ we have $\psi(D) \in C^{(-k)} \cap \mathcal{D}^0$. Since the latter set has $(2^{2kN} - 2)^n$ elements while their (disjoint) union has $2^{2kNn}$, it is easily seen that this is always possible. Thus we obtain a rigid permutation $\psi$ with $\psi(D) \subset D^{(k)}$ for all $D \in \mathcal{D}$ and $\psi(\mathcal{D}^1) \subset \mathcal{D}^0$.

**Remark 4.11.** It is readily observed that

$$T_\phi^\alpha \Pi_\phi^{01} = T_\psi^\alpha T_{\psi\circ\phi}^0 \Pi_{\psi\circ\phi}^{00=} \Pi_\phi^{01}, \qquad T_\phi^\alpha \Pi_\phi^{10} = T_{\phi\circ\psi}^\alpha \Pi_{\phi\circ\psi}^{00=} T_\psi^0 \Pi_\phi^{10},$$
$$T_\phi^\alpha \Pi_\phi^{11} = T_\psi^\alpha T_{\psi\circ\phi\circ\psi}^0 \Pi_{\psi\circ\phi\circ\psi}^{00=} T_\psi^0 \Pi_\phi^{11},$$

where $\Pi_\varphi^{00=} := \Pi_\varphi^{00} + \Pi_\varphi^{=}$ for any permutation $\varphi$.

A straightforward but somewhat tedious computation also shows that

$$\begin{aligned}
U_\phi^\alpha \Pi_\phi^{01} &= \big(U_\psi^\alpha T_{\psi\circ\phi}^0 \Pi_{\psi\circ\phi}^{00=} + U_{\psi\circ\phi}^0 \Pi_{\psi\circ\phi}^{00=} B_1\big)\Pi_\phi^{01},\\
U_\phi^\alpha \Pi_\phi^{10} &= \big(U_{\phi\circ\psi}^\alpha \Pi_{\phi\circ\psi}^{00=} T_\psi^0 + U_\psi^0 B_2\big)\Pi_\phi^{10},\\
U_\phi^\alpha \Pi_\phi^{11} &= \big(U_\psi^\alpha T_{\psi\circ\phi\circ\psi}^0 \Pi_{\psi\circ\phi\circ\psi}^{00=} T_\psi^0 + U_{\psi\circ\phi\circ\psi}^0 \Pi_{\psi\circ\phi\circ\psi}^{00=} T_\psi^0 B_1 + U_\psi^0 B_2\big)\Pi_\phi^{11},
\end{aligned}$$

where the operators $B_1$ and $B_2$ are defined by

$$B_1 : h_{\mathrm{cb}(D,0)}^1 \mapsto \left(\frac{\beta_{\mathrm{cb}(\phi(D),\alpha)}}{\beta_{\mathrm{cb}(\psi\circ\phi(D),0)}}\right)^{1/2} h_{\mathrm{cb}(D,0)}^1,$$
$$B_2 : h_{\mathrm{cb}(D,0)}^1 \mapsto \left(\frac{\beta_{\mathrm{cb}(\phi(D),\alpha)}}{\beta_{\mathrm{cb}(\psi(D),0)}}\right)^{1/2} h_{\mathrm{cb}(D,0)}^1.$$

The uniform boundedness of these operators is plain from Proposition 3.6 and rigidity of $\phi$ combined with the estimates $|\beta_Q| \approx |Q| \approx |\mathrm{dy}(Q)|$ valid for all $Q \in \mathcal{Q}$.

**Remark 4.12.** For a rigid permutation $\varphi$ of $\mathcal{D}$ with the property that $\varphi(D) \subset D^{(k)}$ for every $D \in \mathcal{D}$ the operators $T_\varphi^\alpha \Pi_a^i$ and $U_\varphi^\alpha \Pi_a^i$ are bounded with norms independent of $\varphi$ and $k$ (which defines $a$).

In fact, we can use the same martingale transforms as in Lemmata 4.8 and 4.10, and the required compatibility conditions are ensured by Remark 4.3.

With the lemmata and remarks stated above, the following proposition, which generalizes Figiel's Proposition 3.16 (and his corresponding result for $\hat{U}_\phi$, cf. [10, Theorem 1]) to the "twisted" situation, finally falls to our hands:

**Proposition 4.13.** *Let $\phi$ be a rigid permutation of $\mathcal{D}$ with the property* (3.17) *for some $k \in \mathbf{N}$. Then the operators $T^{\alpha}_{\phi}$ and $U^{\alpha}_{\phi}$ are bounded on $L^p_X(\mathbf{R}^n)$ for $X$ a UMD-space and $1 < p < \infty$, and more precisely*

$$\|T^{\alpha}_{\phi}\|_{\mathcal{L}(L^p_X(\mathbf{R}^n))} \leqslant C(2+k)^{1/r-1/q}, \qquad \|U^{\alpha}_{\phi}\|_{\mathcal{L}(L^p_X(\mathbf{R}^n))} \leqslant C(2+k)^{1-1/q}$$

*provided that $L^p_X(\mathbf{R}^n)$ has type $r$ and cotype $q$.*

**Proof.** The point is that the assertion holds with a uniform (in $k$) bound $C(p,n,X)$ for the projections $T^{\alpha}_{\phi}\Pi^i_a$ and $U^{\alpha}_{\phi}\Pi^i_a$ in place of $T^{\alpha}_{\phi}$ and $U^{\alpha}_{\phi}$. Indeed, first, we already know the uniform boundedness of the compositions of these operators with $\Pi^{=}_{\phi}$ or with $\Pi^{00}_{\phi}$. Second, Remark 4.11 shows that the remaining compositions with $\Pi^{\epsilon\eta}_{\phi}$ for $(\epsilon,\eta) \neq (0,0)$ are expressed in terms of operators of the kind first mentioned, plus the bounded operators $B_i$ and the ones mentioned in Remark 4.12.

Now the assertion of the proposition with exponent 1 in place of $1/r - 1/q$ follows at once from the triangle inequality, for there are $2+k$ different projections $\Pi^i_a$, $i \in \mathbf{Z}_{k+2}$, which sum up to the identity. This argument is quite easily refined to obtain the asserted exponent; the procedure is explained in Figiel [10, Lemma 1]. □

By the observations at the beginning of the present section, we also immediately obtain:

**Corollary 4.14.** *The previous proposition also holds with $T^{\kappa}_{\phi}$ and $U^{\kappa}_{\phi}$ in place of $T^{\alpha}_{\phi}$ and $U^{\alpha}_{\phi}$.*

This completes our investigation of the permutations of our twisted bases.

## 5. Decomposition into elementary operators

Our next goal is to show that rather general operators can be expressed as infinite linear combinations of the elementary operators $T^{\kappa}_m$ and $U^{\kappa}_m$, which we investigated in the previous section. Let us start by describing the general set-up. As usual, we are given a linear operator acting on a restricted class of test functions, and we would like to extend its action to the whole space $L^p_X(\mathbf{R}^n)$, even showing that the image always lies in $L^p_Y(\mathbf{R}^n)$.

To describe our class of test functions, let us first define

$$\begin{aligned}\Xi := \big\{(\theta,\tilde{\theta},Q,\tilde{Q})\colon\ & Q \in \mathcal{Q},\ \tilde{Q} \in \tilde{\mathcal{Q}},\ \operatorname{gen}(Q) = \operatorname{gen}(\tilde{Q}),\\ & (\theta,\tilde{\theta}) \in \mathbf{Z}_{d_Q} \times \mathbf{Z}_{\tilde{d}_{\tilde{Q}}} \setminus \{(0,0)\}\big\},\end{aligned}$$

and then

$$\mathcal{H} := \big\{h^{\theta}_Q \otimes \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}\colon\ (\theta,\tilde{\theta},Q,\tilde{Q}) \in \Xi\big\}.$$

Observe that these are functions on $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$! The best way to gain intuition into this set is probably to consider the case that $n = 1$ and both systems of functions, $h_Q^\theta$ and $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$, coincide with the Haar system in $\mathbf{R}$. Then $\mathcal{H}$ is simply the Haar system in $\mathbf{R}^2$; see [11].

The requirement that $\operatorname{gen}(Q) = \operatorname{gen}(\tilde{Q})$ implies that the supports of the functions in $\mathcal{H}$ are essentially "cubes" in the sense that their dimensions are comparable in all coordinate directions. Nevertheless, we observe that $h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^\theta \in \operatorname{span}\mathcal{H}$ for all $Q \in \mathcal{Q}$, $\tilde{Q} \in \mathcal{Q}$ and $\theta \neq 0 \neq \theta$.

Indeed, this is clear if $\operatorname{gen}(Q) = \operatorname{gen}(\tilde{Q})$, and otherwise we may assume by symmetry that $\operatorname{gen}(Q) < \operatorname{gen}(\tilde{Q}) =: j$. But then

$$h_Q^\theta = \sum_{R \in \mathcal{Q}_j \cap \operatorname{supp} h_Q^\theta} \frac{h_Q^\theta(R)}{h_R^0(R)} \cdot h_R^0 = \sum_{R \in \mathcal{Q}_j \cap Q} \langle h_Q^\theta, h_R^0 \rangle_b \cdot h_R^0 \tag{5.1}$$

where $h_Q^\theta(R)$ (respectively $h_R^0(R)$) denotes the constant non-zero value of $h_Q^\theta$ (respectively $h_R^0(R)$) on $R$, and from this the assertion is clear.

Our initial operator is a linear mapping $\mathfrak{t} : \operatorname{span}\mathcal{H} \to \mathcal{L}(X, Y)$. Our goal is to prove the existence of $T \in \mathcal{L}(L_X^p(\mathbf{R}^n), L_Y^p(\mathbf{R}^n))$ such that

$$\langle y' \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}, M_{\tilde{b}} T M_b (x \otimes h_Q^\theta) \rangle = \langle y', \mathfrak{t}(h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}) x \rangle \tag{5.2}$$

for all $x \in X$, $y' \in Y'$ and $(\theta, \tilde{\theta}, Q, \tilde{Q}) \in \Xi$. Here $M_b$ designates the multiplication operator by the function $b$. To facilitate writing, we also denote $t(\theta, \tilde{\theta}, Q, \tilde{Q}) := \mathfrak{t}(h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}})$ for $(\theta, \tilde{\theta}, Q, \tilde{Q}) \in \Xi$.

Observe that for $b, b^{-1} \in L^\infty(\mathbf{R}^n)$ the operator $M_b$, acting on the Banach spaces $L_X^p(\mathbf{R}^n)$, is self-adjoint (in the sense of Banach adjoints) and invertible; in fact $M_b^{-1} = M_{b^{-1}}$. Moreover,

$$\langle g, M_{\tilde{b}} T M_b f \rangle = \langle g, T M_b f \rangle_{\tilde{b}} = \langle T' M_{\tilde{b}} g, f \rangle_b \tag{5.3}$$

for $f \in L_X^p(\mathbf{R}^n)$, $g \in L_{Y'}^{p'}(\mathbf{R}^n)$.

To begin with, we seek for operators $T$ and $T'$ such that, for every $y' \in Y'$ and $x \in X$,

$$y' \circ T : X \otimes M_b \operatorname{span}\big(h_Q^\theta\big)_{Q \in \mathcal{Q}}^{1 \leqslant \theta < d_Q} \to L^p\big(\mathbf{R}^n\big),$$

$$x \circ T' : Y' \otimes M_{\tilde{b}} \operatorname{span}\big(\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}\big)_{\tilde{Q} \in \tilde{\mathcal{Q}}}^{1 \leqslant \tilde{\theta} < \tilde{d}_{\tilde{Q}}} \to L^{p'}\big(\mathbf{R}^n\big),$$

and the functions $T M_b (x \otimes h_Q^\theta)$ and $T' M_{\tilde{b}} (y' \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}})$ are strongly measurable.

We further require that $T$ and $T'$ are *formally adjoint* in the following sense: (5.3) should hold whenever

$$f = x \otimes h_Q^\theta, \qquad g = y' \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}} \quad \text{and} \quad \theta \neq 0 \neq \tilde{\theta}.$$

To relate these operators to $t$ (or t), we require for $h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}} \in \mathcal{H}$ the equalities

$$\langle y', t(\theta, \tilde{\theta}, Q, \tilde{Q})x\rangle = \langle y' \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}, TM_b(x \otimes h_Q^\theta)\rangle_{\tilde{b}} \quad \text{if } \theta \neq 0,$$
$$\langle y', t(\theta, \tilde{\theta}, Q, \tilde{Q})x\rangle = \langle T'M_{\tilde{b}}(y' \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}), x \otimes h_Q^\theta\rangle_b \quad \text{if } \tilde{\theta} \neq 0.$$

We say that $(T, T')$ with these properties is the *operator pair associated with $t$*. The following remark justifies the use of the definite article.

**Remark 5.4.** There is at most one operator pair associated with $t$.

**Proof.** By (5.1) we find that the value of the pairing

$$\langle y' \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}, TM_b(x \otimes h_Q^\theta)\rangle_{\tilde{b}} \quad \big(\text{respectively } \langle T'M_{\tilde{b}}(y' \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}), x \otimes h_Q^\theta\rangle_b\big)$$

is uniquely determined by $t$ for $\theta \neq 0 \neq \tilde{\theta}$ and $\operatorname{gen}(\tilde{Q}) \leqslant \operatorname{gen}(Q)$ (respectively $\operatorname{gen}(\tilde{Q}) \geqslant \operatorname{gen}(Q)$). But then the requirement of formal adjointness of $T$ and $T'$ provides the same conclusion for all $Q \in \mathcal{Q}$ and $\tilde{Q} \in \tilde{\mathcal{Q}}$.

For every $y' \in Y'$, we have $y'TM_b(x \otimes h_Q^\theta) \in L^p(\mathbf{R}^n)$. By Proposition 3.6, this $L^p$-function can be recovered from its $\tilde{b}$-pairings with all the $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$, which were just seen to be determined. Thus $y'TM_b(x \otimes h_Q^\theta)$ is uniquely determined as an element of $L^p(\mathbf{R}^n)$ and hence almost everywhere. Since $TM_b(x \otimes h_Q^\theta)$ is strongly measurable, whence essentially separably valued, countable many $y' \in Y'$ suffice for its determination a.e., and thus the values of the function $TM_b(x \otimes h_Q^\theta)$ are a.e. uniquely determined by $t$. A symmetric reasoning applies to $T'M_{\tilde{b}}(y' \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}})$. Thus $T$ and $T'$ are determined on their respective ranges in terms of $t$. $\square$

To facilitate the identification of the pairs of operators associated with $t$, we consider three basic types of functions $t$ into which any $t$ can be split in a canonical manner. We say that $t$ is of type $(\epsilon, \eta)$ if $t(\theta, \tilde{\theta}, Q, \tilde{Q}) \neq 0$ implies $(\theta, \tilde{\theta}) \in \Theta_{\epsilon\eta}$, where $\Theta_{01} := \{0\} \times \mathbf{Z}'_d$, $\Theta_{10} := \mathbf{Z}'_d \times \{0\}$, and $\Theta_{11} := \mathbf{Z}'_d \times \mathbf{Z}'_d$; recall that $\mathbf{Z}'_d := \mathbf{Z}_d \setminus \{0\}$.

**Lemma 5.5.** *If $t$ is of type $(1, 1)$, then a pair of formal adjoints $T, T'$ is associated with $t$ if and only if, for all $Q \in \mathcal{Q}$, $1 \leqslant \theta < d_Q$,*

$$TM_b(x \otimes h_Q^\theta) = \sum_{\tilde{Q} \in \tilde{\mathcal{Q}}_{\operatorname{gen}(Q)}} \sum_{1 \leqslant \tilde{\theta} < \tilde{d}_{\tilde{Q}}} t(\theta, \tilde{\theta}, Q, \tilde{Q})x \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}.$$

**Proof.** Let $T$ be given by the above formula, and let $h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}} \in \mathcal{H}$. If $\theta \neq 0$, then

$$\langle y' \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}, TM_b(x \otimes h_Q^\theta)\rangle_{\tilde{b}} = \sum_{1 \leqslant \tilde{\vartheta} < \tilde{d}_{\tilde{Q}}} \langle y', t(\theta, \tilde{\vartheta}, Q, \tilde{Q})x\rangle \langle \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}, \tilde{h}_{\tilde{Q}}^{\tilde{\vartheta}}\rangle_{\tilde{b}} = \langle y', t(\theta, \tilde{\theta}, Q, \tilde{Q})x\rangle$$

(which, of course, is zero if $\tilde{\theta} = 0$), and if $\tilde{\theta} \neq 0$, then

$$\langle T' M_{\tilde{b}}(y' \otimes \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}), x \otimes h^0_Q \rangle_b = \sum_{R \supsetneq Q} \sum_{1 \leqslant \theta < d_R} \langle T' M_{\tilde{b}}(y' \otimes \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}), x \otimes h^\theta_R \rangle_b \langle h^0_Q, h^\theta_R \rangle_b.$$

But the first factor in the summand is equal to

$$\langle y' \otimes \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}, T M_b(x \otimes h^\theta_R) \rangle_{\tilde{b}} = \sum_{\tilde{R} \in \tilde{\mathcal{Q}}_{\mathrm{gen}(R)}} \langle y' \otimes \tilde{h}^0_{\tilde{R}}, T M_b(x \otimes h^\theta_R) \rangle_{\tilde{b}} \langle \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}, \tilde{h}^0_{\tilde{R}} \rangle_{\tilde{b}},$$

and here the second factor of each summand vanishes, since $\langle \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}, \tilde{h}^0_{\tilde{R}} \rangle_{\tilde{b}} \neq 0$ implies $\tilde{R} \subset \tilde{Q}$, which cannot be the case as $\mathrm{gen}(\tilde{R}) = \mathrm{gen}(R) < \mathrm{gen}(Q) = \mathrm{gen}(\tilde{Q})$.

Conversely, let $T, T'$ be associated with $t$. Then $y' T M_b(x \otimes h^\theta_Q) \in L^p(\mathbf{R}^n)$ can be expanded as

$$y' T M_b(x \otimes h^\theta_Q) = \sum_{\tilde{Q} \in \tilde{\mathcal{Q}}} \sum_{1 \leqslant \tilde{\theta} < \tilde{d}_{\tilde{Q}}} \langle y' \otimes \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}, T M_b(x \otimes h^\theta_Q) \rangle_{\tilde{b}} \tilde{h}^{\tilde{\theta}}_{\tilde{Q}} \tag{5.6}$$

with unconditional convergence in $L^p(\mathbf{R}^n)$ (by Proposition 3.6 again). If $\mathrm{gen}(\tilde{Q}) < \mathrm{gen}(Q)$, an expansion of $\tilde{h}^{\tilde{\theta}}_{\tilde{Q}}$ in terms of $\tilde{h}^0_{\tilde{R}}$ reveals that the corresponding summand above must vanish. A symmetric reasoning and an application of formal duality gives the same conclusion for $\mathrm{gen}(\tilde{Q}) > \mathrm{gen}(Q)$. Thus only terms with $\tilde{Q} \in \tilde{\mathcal{Q}}_{\mathrm{gen}(Q)}$ survive, and then the conclusion requires nothing but a standard application of the Hahn–Banach theorem. □

**Lemma 5.7.** *If $t$ is of type $(1, 0)$, then a pair of formal adjoints $T$, $T'$ is associated with $t$ if and only if, for all $Q \in \mathcal{Q}$ and $1 \leqslant \theta < d_Q$,*

$$T M_b(x \otimes h^\theta_Q) = \sum_{\tilde{Q} \in \tilde{\mathcal{Q}}_{\mathrm{gen}(Q)}} t(\theta, 0, Q, \tilde{Q}) x \otimes \tilde{h}^0_{\tilde{Q}}.$$

**Proof.** If $T$ is given by the above formula, then the verification that $T$ and $T'$ are associated with $t$ is essentially the same as in the previous proof.

For the converse, we again use the $L^p(\mathbf{R}^n)$-unconditional expansion (5.6). This time, the vanishing values of $t$ imply the vanishing of $\langle y' \otimes \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}, T M_b(x \otimes h^\theta_Q) \rangle_{\tilde{b}}$ for $\mathrm{gen}(\tilde{Q}) \geqslant \mathrm{gen}(Q)$. Thus we are left with

$$\begin{aligned}
&y' T M_b(x \otimes h^\theta_Q) \\
&\quad = \lim_{k \to -\infty} \sum_{k < \mathrm{gen}(\tilde{Q}) < \mathrm{gen}(Q)} \sum_{1 \leqslant \tilde{\theta} < \tilde{d}_{\tilde{Q}}} \langle y' \otimes \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}, T M_b(x \otimes h^\theta_Q) \rangle_{\tilde{b}} \tilde{h}^{\tilde{\theta}}_{\tilde{Q}} \\
&\quad = \lim \sum_{\tilde{R} \in \tilde{\mathcal{Q}}_{\mathrm{gen}(Q)}} \tilde{h}^0_{\tilde{R}} \sum_{\tilde{Q} \supsetneq \tilde{R},\ \mathrm{gen}(\tilde{Q}) > k} \sum_{1 \leqslant \tilde{\theta} < \tilde{d}_{\tilde{Q}}} \langle \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}, \tilde{h}^0_{\tilde{R}} \rangle_{\tilde{b}} \langle y' \otimes \tilde{h}^{\tilde{\theta}}_{\tilde{Q}}, T M_b(x \otimes h^\theta_Q) \rangle_{\tilde{b}}.
\end{aligned}$$

We are assuming that $k$ runs over an appropriate subsequence for which pointwise a.e. convergence holds in the first place; then we used the expansion of $\tilde h^{\tilde\theta}_{\tilde Q}$ and the fact that for a fixed $k$ the summations contain locally only finitely many non-zero terms which justifies the exchange of their order. Next, we bring the a.e. limit inside the first summation (which is easily justified by the disjointess of the $\tilde h^0_{\tilde R}$) to continue the previous chain of equalities with

$$\begin{aligned}&= \sum_{\tilde R\in\tilde{\mathcal{Q}}_{\mathrm{gen}(Q)}} \tilde h^0_{\tilde R} \sum_{\tilde Q\supsetneq\tilde R}\sum_{1\leqslant\tilde\theta<\tilde d_{\tilde Q}} \langle \tilde h^{\tilde\theta}_{\tilde Q}, \tilde h^0_{\tilde R}\rangle_{\tilde b}\langle y'\otimes\tilde h^{\tilde\theta}_{\tilde Q}, TM_b\big(x\otimes h^\theta_Q\big)\rangle_{\tilde b}\\
&= \sum_{\tilde R\in\tilde{\mathcal{Q}}_{\mathrm{gen}(Q)}} \tilde h^0_{\tilde R}\langle y'\otimes\tilde h^0_{\tilde R}, TM_b\big(x\otimes h^\theta_Q\big)\rangle_{\tilde b},\end{aligned}$$

and Hahn–Banach theorem complete the argument. □

By reasons of symmetry, the following result is now clear, too.

**Corollary 5.8.** *If $t$ is of type $(0,1)$, then a pair of formal adjoints $T$, $T'$ is associated with $t$ if and only if, for all $\tilde Q\in\tilde{\mathcal{Q}}$ and $1\leqslant\tilde\theta<\tilde d_{\tilde Q}$,*

$$T'M_{\tilde b}\big(y'\otimes\tilde h^{\tilde\theta}_{\tilde Q}\big) = \sum_{Q\in\mathcal{Q}_{\mathrm{gen}(\tilde Q)}} t(0,\tilde\theta,Q,\tilde Q)'y'\otimes h^0_Q.$$

We now decompose an arbitrary $t$ into parts of the three different types in the following obvious fashion:

$$t^{\epsilon\eta}(\theta,\tilde\theta,Q,\tilde Q) := 1_{\Theta_{\epsilon\eta}}(\theta,\tilde\theta)t(\theta,\tilde\theta,Q,\tilde Q).$$

Furthermore, for $\kappa=(\alpha,\tilde\alpha,\vartheta,\tilde\vartheta)$ and $m\in\mathbf{Z}^n$, we let

$$t^{\epsilon\eta}_{\kappa,m}(\theta,\tilde\theta,Q,\tilde Q) = \delta_{\theta\vartheta}\delta_{\tilde\theta\tilde\vartheta}\delta_{\mathrm{tp}(Q),\alpha}\delta_{\mathrm{tp}(\tilde Q),\tilde\alpha}\delta_{\mathrm{dy}(Q)\dot+m,\mathrm{dy}(\tilde Q)}t^{\epsilon\eta}(\theta,\tilde\theta,Q,\tilde Q),$$

where $D\dot+m := D+m\cdot\ell(D)$; here $\ell(D)$ is the side length of the dyadic cube $D$. Note that $D\mapsto D\dot+m$ is a rigid permutation of $\mathcal{D}$; with slight misuse of notation, we also denote this permutation by $m$.

The part of type $(1,1)$ is most easily identified with a bounded operator under an easily stated condition. Obviously

$$t^{11} = \sum_{\kappa\in(\mathbf{Z}_2)^2\times(\mathbf{Z}'_d)^2}\sum_{m\in\mathbf{Z}^n} t^{11}_{\kappa,m},$$

and the functions $t^{11}_{\kappa,m}$ are supported on disjoint subsets of $\Xi$.

We make the following $R$-boundedness assumption on these pieces of $t$:

$$\begin{aligned} r^{1\eta}_{\kappa,m} &:= \operatorname*{ess\,sup}_{\xi\in\mathbf{R}^n} \mathcal{R}\big(\big\{t^{1\eta}_{\kappa,m}(\theta,\tilde\theta,Q,\tilde Q)\colon (\theta,\tilde\theta,Q,\tilde Q)\in\Xi,\ Q\ni\xi\big\}\big)<\infty, \\ &\quad \text{for } \eta\in\{0,1\}, \quad \text{and} \\ r^{01}_{\kappa,m} &:= \operatorname*{ess\,sup}_{\xi\in\mathbf{R}^n} \mathcal{R}\big(\big\{t^{01}_{\kappa,m}(\theta,\tilde\theta,Q,\tilde Q)\colon (\theta,\tilde\theta,Q,\tilde Q)\in\Xi,\ \tilde Q\ni\xi\big\}\big)<\infty. \end{aligned} \tag{5.9}$$

Note that these requirements are somewhat weaker than the simpler condition

$$\mathcal{R}\big(\operatorname{range} t^{\epsilon\eta}_{\kappa,m}\big)<\infty.$$

**Lemma 5.10.** $t^{11}_{\kappa,m}$ *is associated with the operator*

$$T^\kappa_m \Lambda^\kappa_m M_b^{-1}\in\mathcal{L}\big(L^p_X\big(\mathbf{R}^n\big),L^p_Y\big(\mathbf{R}^n\big)\big)$$

(*and its usual adjoint*), *where*

$$\Lambda^\kappa_m : x\otimes h^{\vartheta}_{\mathrm{cb}(D,\alpha)}\mapsto t\big(\vartheta,\tilde\vartheta,\mathrm{cb}(D,\alpha),\tilde{\mathrm{c}}\mathrm{b}(D\dot{+}m,\tilde\alpha)\big)x\otimes h^{\vartheta}_{\mathrm{cb}(D,\alpha)},$$

*and all other* $h^\theta_Q$ *are mapped into zero. We have*

$$\big\|\Lambda^\kappa_m\big\|_{\mathcal{L}(L^p_X(\mathbf{R}^n),L^p_Y(\mathbf{R}^n))}\leqslant Cr^{11}_{\kappa,m}.$$

**Proof.** The norm estimate for $\Lambda^\kappa_m$ is plain from Proposition 3.6 and the definition of $R$-boundedness. The rest follows from Lemma 5.5. □

**Corollary 5.11.** *If the series* $\sum_{\kappa,m} T^\kappa_m\Lambda^\kappa_m M_b^{-1}$ *converges absolutely in the norm of* $\mathcal{L}(L^p_X(\mathbf{R}^n),L^p_Y(\mathbf{R}^n))$, *then its sum is the operator associated with* $t^{11}$. *The convergence holds, in particular, if*

$$\sum_{\kappa,m}\log^{1/r-1/q}\big(2+|m|\big)r^{11}_{\kappa,m}<\infty,$$

*where* $L^p_Y(\mathbf{R}^n)$ *has type* $r$ *and cotype* $q$.

**Proof.** This is immediate from the previous lemma and Proposition 4.13, which provides the norm estimate

$$\big\|T^\kappa_m\big\|_{\mathcal{L}(L^p_Y(\mathbf{R}^n))}\leqslant C\log^{1/r-1/q}\big(2+|m|\big). \qquad \square$$

We note in passing that one could also easily give an explicit description of the adjoint of the operator associated with $t^{11}_{\kappa,m}$, and then obtain, as above, a criterion of convergence for the adjoint series in terms of the type and cotype of $L^{p'}_{X'}(\mathbf{R}^n)$.

Then we turn to $t^{10}$; symmetric reasoning will apply to $t^{01}$. Let us introduce some auxiliary functions. Denoting $D := \mathrm{dy}(Q)$, we set

$$p^{10}_{\kappa,m}(\theta,\tilde\theta,Q,\tilde Q) := \delta_{\theta\vartheta}\delta_{\tilde\theta 0}\delta_{\mathrm{tp}(Q),\alpha}\delta_{\mathrm{tp}(\tilde Q),0}\delta_{D,\,\mathrm{dy}(\tilde Q)} \times t^{10}\big(\theta,0,Q,\tilde{\mathrm{c}}\mathrm{b}(D\dotplus m,\tilde\alpha)\big)\left(\frac{\tilde\beta_{\tilde{\mathrm{c}}\mathrm{b}(D\dotplus m,\tilde\alpha)}}{\tilde\beta_{\tilde{\mathrm{c}}\mathrm{b}(D,0)}}\right)^{1/2}, \tag{5.12}$$

$$u^{10}_{\kappa,m} := t^{10}_{\kappa,m} - p^{10}_{\kappa,m}, \qquad u^{10} := \sum_{\kappa,m} u^{10}_{\kappa,m}, \qquad p^{10}_{\alpha\theta} := \sum_{\tilde\alpha,m} p^{10}_{(\alpha,\tilde\alpha,\theta,0),m}$$

A word on the convergence of these series is in order. It is readily seen that the pointwise convergence of $p^{10}_{\alpha\theta}$ (as a function on $\Xi$) requires the convergence of the series

$$p^\theta_Q := \sum_{\tilde Q\in\tilde{\mathcal Q}_{\mathrm{gen}(Q)}} t^{10}(\theta,0,Q,\tilde Q)\tilde\beta^{1/2}_{\tilde Q}$$

for all $Q\in\mathcal Q$ with $\mathrm{tp}(Q)=\alpha$ and $d_Q>\theta$. Clearly

$$\big\|t^{10}\big(\theta,0,\mathrm{cb}(D,\alpha),\tilde{\mathrm{c}}\mathrm{b}(D\dotplus m,\tilde\alpha)\big)\big\|_{\mathcal L(X,Y)} \leqslant r^{10}_{(\alpha,\tilde\alpha,\theta,0),m}.$$

Thus a sufficient condition for the norm convergence of $p^\theta_Q$ is $\sum_{\kappa,m} r^{10}_{\kappa,m}<\infty$. This we will assume, and, in fact, a little more in Lemma 5.13. Thus all the series above are convergent in the norm of $\mathcal L(X,Y)$, pointwise on $\Xi$.

The functions that we have denoted by $u^{10}_{\kappa,m}$ are handled in a fashion analogous to the $t^{11}_{\kappa,m}$, only resorting to Lemma 5.7 in place of Lemma 5.5. This leads to:

**Lemma 5.13.** *$u^{10}_{\kappa,m}$ is associated with the operator*

$$U^\kappa_m\Lambda^\kappa_m M_b^{-1}\in\mathcal L\big(L^p_X(\mathbf R^n),L^p_Y(\mathbf R^n)\big).$$

*If the series $\sum_{\kappa,m} U^\kappa_m\Lambda^\kappa_m M_b^{-1}$ converges, in particular if $L^p_Y(\mathbf R^n)$ has cotype $q$ and $\sum_{\kappa,m}\log^{1-1/q}(2+|m|)r^{10}_{\kappa,m}<\infty$, then its sum is associated with $u^{10}$.*

The analogous "$U$ part" of $t^{01}$ is handled similarly: with $u^{01}_{\kappa,m}$ and $p^{01}_{\tilde\alpha\tilde\theta}$ defined in obvious analogy to (5.12), we get

**Corollary 5.14.** *$u^{01}_{\kappa,m}$ is associated with the adjoint of*

$$\tilde U^\kappa_{-m}\tilde\Lambda^\kappa_m M_{\tilde b}^{-1}\in\mathcal L\big(L^p_{Y'}(\mathbf R^n),L^p_{X'}(\mathbf R^n)\big),$$

*where*

$$\tilde{\Lambda}^{\kappa}_{m}: y'\otimes\tilde{h}^{\tilde{\vartheta}}_{\tilde{\mathrm{c}}\mathrm{b}(D,\tilde{\alpha})}\mapsto t\big(0,\tilde{\vartheta},\mathrm{cb}(D\dot{-}m,\alpha),\tilde{\mathrm{c}}\mathrm{b}(D,\tilde{\alpha})\big)'y'\otimes\tilde{h}^{\tilde{\vartheta}}_{\tilde{\mathrm{c}}\mathrm{b}(D,\tilde{\alpha})},$$

$$\tilde{U}^{\kappa}_{-m}: \tilde{h}^{\tilde{\vartheta}}_{\tilde{\mathrm{c}}\mathrm{b}(D,\tilde{\alpha})}\mapsto h^{0}_{\mathrm{cb}(D\dot{-}m,\alpha)}-\left(\beta_{\mathrm{cb}(D\dot{-}m,\alpha)}/\beta_{\mathrm{cb}(D,0)}\right)^{1/2}h^{0}_{\mathrm{cb}(D,0)}.$$

*If the series $\sum_{\kappa,m}\tilde{U}^{\kappa}_{-m}\tilde{\Lambda}^{\kappa}_{m}M^{-1}_{\tilde{b}}$ converges, in particular if $L^p_X(\mathbf{R}^n)$ has type $r$ (hence $L^{p'}_{X'}(\mathbf{R}^n)$ cotype $r'$) and $\sum_{\kappa,m}\log^{1/r}(2+|m|)r^{01}_{\kappa,m}<\infty$, then its sum is the adjoint of the operator associated with $u^{01}$.*

The functions $p^{10}_{\alpha\theta}$ and $p^{01}_{\tilde{\alpha}\tilde{\theta}}$, on the other hand, lead to a rather different kind of investigation, which we next take up.

## 6. Paraproducts again, and the abstract $Tb$ theorem

By Lemma 5.7, operator $P^{10}_{\alpha\theta}$ associated with $p^{10}_{\alpha\theta}$ should be given by the formula

$$P^{10}_{\alpha\theta}M_b: x\otimes h^{\vartheta}_{Q}\mapsto\delta_{\theta\vartheta}\delta_{\alpha,\mathrm{tp}(Q)}p^{\theta}_{Q}x\otimes\tilde{\beta}^{-1/2}_{\tilde{\mathrm{c}}\mathrm{b}(\mathrm{dy}(Q),0)}\tilde{h}^{0}_{\tilde{\mathrm{c}}\mathrm{b}(\mathrm{dy}(Q),0)}.$$

Our purpose, again, is to identify this with a combination of the operators of the elementary types. The principal part will now be a paraproduct operator.

Let us start with a simple factorization of our transformation. We know already the existence of the bounded mapping $T^{(\alpha,0,\theta,1)}_{0}$ sending $x\otimes h^{\theta}_{\mathrm{cb}(D,\alpha)}\mapsto x\otimes\tilde{h}^{1}_{\tilde{\mathrm{c}}\mathrm{b}(D,0)}$. To produce $P^{10}_{\alpha\theta}M_b$, we need to combine this with an application of

$$\mathcal{P}^{10}_{\alpha\theta}M_{\tilde{b}}: x\otimes\tilde{h}^{1}_{\tilde{Q}}\mapsto\delta_{\mathrm{tp}(\tilde{Q}),0}p^{\theta}_{\mathrm{cb}(\mathrm{dy}(\tilde{Q}),\alpha)}\otimes\tilde{\beta}^{-1/2}_{\tilde{Q}}\tilde{h}^{0}_{\tilde{Q}}.$$

Note that $\tilde{\beta}^{-1/2}_{\tilde{Q}}\tilde{h}^{0}_{\tilde{Q}}=\tilde{\beta}^{-1}_{\tilde{Q}}1_{\tilde{Q}}$.

We can now compute

$$\big\langle\big(\mathcal{P}^{10}_{\alpha\theta}\big)'M_{\tilde{b}}\big(y'\otimes\tilde{h}^{\tilde{\theta}}_{\tilde{R}}\big),x\otimes\tilde{h}^{\tilde{\vartheta}}_{\tilde{Q}}\big\rangle_{\tilde{b}}=\delta_{\tilde{\vartheta},1}\delta_{\mathrm{tp}(\tilde{Q}),0}\big\langle y',p^{\theta}_{\mathrm{cb}(\mathrm{dy}(\tilde{Q}),\alpha)}x\big\rangle\big\langle\tilde{h}^{\tilde{\theta}}_{\tilde{R}},\tilde{\beta}^{-1}_{\tilde{Q}}1_{\tilde{Q}}\big\rangle_{\tilde{b}},$$

and the last pairing equals $\tilde{h}^{\tilde{\theta}}_{\tilde{R}}(\tilde{Q})$ if $\tilde{Q}\subsetneq\tilde{R}$ and zero, otherwise.

It follows that

$$\begin{aligned}\big\langle\big(\mathcal{P}^{10}_{\alpha\theta}\big)'M_{\tilde{b}}\big(y'\otimes\tilde{h}^{\tilde{\theta}}_{\tilde{R}}\big),x\big\rangle&=\sum_{\tilde{Q}\subsetneq\tilde{R},\,\mathrm{tp}(\tilde{Q})=0}\big\langle\big(p^{\theta}_{\mathrm{cb}(\mathrm{dy}(\tilde{Q}),\alpha)}\big)'y',x\big\rangle\tilde{h}^{\tilde{\theta}}_{\tilde{R}}(\tilde{Q})\cdot\tilde{h}^{1}_{\tilde{Q}}\\&=\tilde{h}^{\tilde{\theta}}_{\tilde{R}}\sum_{\tilde{Q}\subsetneq\tilde{R},\,\mathrm{tp}(\tilde{Q})=0}\big\langle\big(p^{\theta}_{\mathrm{cb}(\mathrm{dy}(\tilde{Q}),\alpha)}\big)'y',x\big\rangle\tilde{h}^{1}_{\tilde{Q}}\\&=\tilde{h}^{\tilde{\theta}}_{\tilde{R}}\cdot\big(\mathrm{id}-\tilde{F}_{\mathrm{gen}(\tilde{R})}\big)\big\langle w_{\alpha\theta}(\cdot)y',x\big\rangle,\end{aligned}\tag{6.1}$$

where $\tilde{F}_j$ denotes the $\tilde{b}$-twisted conditional expectation with respect to the $\sigma$-algebra $\sigma(\tilde{\mathcal{Q}}_j)$, and $w_{\alpha\theta}$ designates the (so far) formal series

$$w_{\alpha\theta} := \sum_{\tilde{Q}\in\tilde{\mathcal{Q}},\, \mathrm{tp}(\tilde{Q})=0} \big(p^{\theta}_{\mathrm{cb}(\mathrm{dy}(\tilde{Q}),\alpha)}\big)' \cdot \tilde{h}^{1}_{\tilde{Q}}. \tag{6.2}$$

As we are about to see, this series is closely related to the object $T'\tilde{b}$ appearing in the very name of our theorem. We wish to find an expression for this object by formally computing its expansion coefficients in terms of the basis $(h^{\theta}_{Q})^{0<\theta<d_Q}_{Q\in\mathcal{Q}}$. To this end, we have

$$\begin{aligned}\big\langle T'(y'\otimes\tilde{b}), x\otimes h^{\theta}_{Q}\big\rangle_b &= \big\langle y'\otimes 1, TM_b\big(x\otimes h^{\theta}_{Q}\big)\big\rangle_{\tilde{b}} \\ &= \sum_{\tilde{Q}\in\tilde{\mathcal{Q}}_{\mathrm{gen}(Q)}} \tilde{\beta}^{1/2}_{\tilde{Q}}\big\langle y'\otimes\tilde{h}^{0}_{\tilde{Q}}, TM_b\big(x\otimes h^{\theta}_{Q}\big)\big\rangle_{\tilde{b}} \\ &= \sum_{\tilde{Q}\in\tilde{\mathcal{Q}}_{\mathrm{gen}(Q)}} \tilde{\beta}^{1/2}_{\tilde{Q}}\big\langle y', t\big(\theta, 0, Q, \tilde{Q}\big)x\big\rangle = \big\langle y', p^{\theta}_{Q}x\big\rangle, \end{aligned} \tag{6.3}$$

and thus, formally,

$$T'\tilde{b} = \sum_{Q\in\mathcal{Q}}\sum_{0<\theta<d_Q} \big(p^{\theta}_{Q}\big)'\cdot h^{\theta}_{Q}. \tag{6.4}$$

We impose on $T'\tilde{b}$ the following condition. First, we require that the operators $(p^{\theta}_{Q})'$ belong not only to $\mathcal{L}(Y', X')$, but to a UMD $R$-space $V \hookrightarrow \mathcal{L}(Y', X')$. Moreover, we demand that the series (6.4) is the expansion of a function in $BMO_{\mathrm{dy},V}(\mathbf{R}^n)$, the $V$-valued dyadic $BMO$ space, i.e., the filtered $BMO$ space corresponding to the dyadic filtration of $\mathbf{R}^n$. (This is a regular filtration, with $B = 2^n$ in (2.1).) Some remarks concerning this space are in order.

**Remark 6.5.** The space $BMO_{\mathrm{dy},V}(\mathbf{R}^n)$ coincides with the martingale $BMO$ space related to the (also regular) filtration $(\sigma(\mathcal{Q}_j))^{\infty}_{-\infty}$.

This follows easily from elementary estimates, using the facts that every $Q \in \mathcal{Q}$ is a union of at most $M$ (say) dyadic cubes $D_i$, each with measure $|D_i| \geqslant c|Q|$, and conversely. Thus the dyadic $BMO$ norm of a function $f$ is equivalent, for any $p \in [1, \infty[$ to

$$\sup_{Q\in\mathcal{Q}} |Q|^{-1/p}\big\|1_Q\big(f - E_{\mathrm{gen}(Q)}f\big)\big\|_p \approx \sup_{Q\in\mathcal{Q}} |Q|^{-1/p}\big\|1_Q\big(f - F_{\mathrm{gen}(Q)}f\big)\big\|_p,$$

where $F_j$ denotes the $b$-twisted conditional expectation; obviously the same thing is true for the $\tilde{b}$-twisted conditional expectation $\tilde{F}_j$. The last equivalence follows from the facts that multiplication by $1_Q$ commutes with both $E_{\mathrm{gen}(Q)}$ and $F_{\mathrm{gen}(Q)}$, and, moreover,

$\mathrm{id}-E_j=(\mathrm{id}-E_j)(\mathrm{id}-F_j)$ and $\mathrm{id}-F_j=(\mathrm{id}-F_j)(\mathrm{id}-E_j)$ are both bounded operators (uniformly in $j$) on $L^p_X(\mathbf{R}^n)$.

**Remark 6.6.** If $V$ is a UMD space, as we are assuming, and $1<p<\infty$, then (3.7) holds for $f\in BMO_{\mathrm{dy},V}(\mathbf{R}^n)$ with unconditional convergence in $L^p_{V,\mathrm{loc}}(K)/V$, for any compact $K\subset\mathbf{R}^n$. Moreover,

$$\|f\|^p_{BMO_{\mathrm{dy}}}\approx\sup_{Q\in\mathcal{Q}}|Q|^{-1}E_\varepsilon\int\limits_Q\left|\sum_{R\subset Q}\sum_{0<\theta<d_R}\varepsilon^\theta_R\langle f,h^\theta_Q\rangle_b h^\theta_Q(\xi)\right|^p_V\mathrm{d}\xi. \tag{6.7}$$

Conversely, if a set of coefficients $c^\theta_Q\in V$ is given, which make the quantity on the right of (6.7) finite when substituted for $\langle f,h^\theta_Q\rangle_b$, then the corresponding series again converges in the described sense to an element of $BMO_{\mathrm{dy},V}(\mathbf{R}^n)$ such that $\langle f,h^\theta_Q\rangle_b=c^\theta_Q$.

This follows rather readily upon application of Proposition 3.6 to the $L^p_V$ functions $1_Q(f-F_{\mathrm{gen}(Q)}f)$ and using the equivalent norm of $BMO_{\mathrm{dy},V}(\mathbf{R}^n)$ provided by the previous remark. (Cf. also [14], where a similar expansion in terms of a regular wavelet basis $(\psi^\theta_Q)$ in place of $(h^\theta_Q)$ is proved for functions in the usual $BMO$ space $BMO_V(\mathbf{R}^n)$.)

**Remark 6.8.** Projections of the type $\Pi h^\theta_Q\in\{0,h^\theta_Q\}$ ($Q\in\mathcal{Q}$, $0<\theta<d_Q$) are bounded on $BMO_{\mathrm{dy},V}(\mathbf{R}^n)$. So are the mappings $T^\kappa_0:h^\vartheta_{\mathrm{cb}(D,\alpha)}\mapsto\tilde{h}^{\tilde{\vartheta}}_{\tilde{\mathrm{cb}}(D,\tilde{\alpha})}$ ($\kappa=(\alpha,\tilde{\alpha},\vartheta,\tilde{\vartheta})$).

These facts, too, follow from the corresponding estimates on the $L^p_V(\mathbf{R}^n)$ spaces, and the expression of the norm of $BMO_{\mathrm{dy},V}(\mathbf{R}^n)$ in terms of the $L^p_V(\mathbf{R}^n)$-norms.

We now return to our investigation of the operator $\mathcal{P}_{\alpha\theta}$. From (6.2) and (6.4) it is clear that

$$w_{\alpha\theta}=T^\kappa_0T'\tilde{b}\in BMO_{\mathrm{dy},V}\big(\mathbf{R}^n\big),\quad\kappa=(\alpha,0,\theta,1).$$

Consider now the following twisted paraproduct operator:

$$\tilde{\mathcal{P}}(w_{\alpha\theta},f)=\sum\tilde{\Delta}_{j+1}w_{\alpha\theta}\cdot\tilde{F}_jf,$$

where $\tilde{F}_j$ and $\tilde{\Delta}_j$ represent the $\tilde{b}$-twisted conditional expectations (with respect to $(\sigma(\tilde{\mathcal{Q}}_j))^\infty_{-\infty}$) and their differences, as defined earlier. It is clear that an application of this operator to a function of the special type $\tilde{h}=\tilde{\Delta}_k\tilde{h}$ gives

$$\tilde{\mathcal{P}}\big(w_{\alpha\theta},\tilde{h}\big)=\Big(\sum_{j\geqslant k}\tilde{\Delta}_{j+1}w_{\alpha\theta}\Big)\tilde{h}=\big(\mathrm{id}-\tilde{F}_k\big)w_{\alpha\theta}\cdot\tilde{h}\quad\text{for }\tilde{h}=\tilde{\Delta}_k\tilde{h}. \tag{6.9}$$

But realizing that every $\tilde{h} = \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$, for $\tilde{Q} \in \tilde{\mathcal{Q}}_k$ and $0 < \tilde{\theta} < \tilde{d}_{\tilde{Q}}$, is exactly a function of this kind, and comparing (6.9) with (6.1), we recognize the operator identity

$$\big(\mathcal{P}_{\alpha\theta}^{10}\big)' M_{\tilde{b}} = \tilde{\mathcal{P}}(w_{\alpha\theta}, \cdot).$$

By virtue of Proposition 2.15, this gives at once the boundedness of $\mathcal{P}_{\alpha\theta}$ as soon as $w_{\alpha\theta} \in BMO_{\mathrm{dy},V}(\mathbf{R}^n)$, which, as we saw, is a consequence of the assumption "$T'\tilde{b} \in BMO_{\mathrm{dy},V}(\mathbf{R}^n)$."

To summarize:

**Lemma 6.10.** *Suppose that $T'\tilde{b}$, defined by its $h_Q^\theta$-coefficients as in* (6.3)*, satisfies $T'\tilde{b} \in BMO_{\mathrm{dy},V}(\mathbf{R}^n)$, where $V \hookrightarrow \mathcal{L}(Y', X')$ is a UMD R-space. Then $p_{\alpha\theta}^{10}$ is associated to the operator $P_{\alpha\theta}^{10} \in \mathcal{L}(L_X^p(\mathbf{R}^n), L_Y^p(\mathbf{R}^n))$ given by*

$$P_{\alpha\theta}^{10} = M_{\tilde{b}}^{-1} \circ \big[\tilde{\mathcal{P}}\big(T_0^\kappa \circ T'\tilde{b}, \cdot\big)\big]' \circ M_{\tilde{b}} \circ T_0^\kappa \circ M_b^{-1}, \quad \kappa = (\alpha, 0, \theta, 1).$$

The argument leading to this lemma is easily modified to yield the case of $p_{\tilde{\alpha}\tilde{\theta}}^{01}$:

**Corollary 6.11.** *Suppose that $Tb$, defined in terms of its $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$-coefficients, belongs to the space $BMO_{U,\mathrm{dy}}(\mathbf{R}^n)$, where $U \hookrightarrow \mathcal{L}(X, Y)$ is a UMD R-space. Then $p_{\tilde{\alpha}\tilde{\theta}}^{01}$ is associated with the operator $P_{\tilde{\alpha}\tilde{\theta}}^{01} \in \mathcal{L}(L_X^p(\mathbf{R}^n), L_Y^p(\mathbf{R}^n))$ given by*

$$P_{\tilde{\alpha}\tilde{\theta}}^{01} = M_{\tilde{b}}^{-1} \circ \big(\tilde{T}_0^\kappa\big)' \circ M_b \circ \mathcal{P}\big(\tilde{T}_0^\kappa \circ Tb, \cdot\big) \circ M_b^{-1}, \quad \kappa = (0, \tilde{\alpha}, 1, \tilde{\theta}),$$

*where $\tilde{T}_0^\kappa : \tilde{h}_{\tilde{\mathrm{c}}\mathrm{b}(D,\tilde{\alpha})}^{\tilde{\theta}} \mapsto h_{\mathrm{cb}(D,0)}^1$.*

Now we merely need to collect everything proved so far to obtain the following abstract formulation of the $Tb$ theorem.

**Theorem 6.12.** *Let $X$ and $Y$ be UMD-spaces and $1 < p < \infty$. Let $1 < r \leqslant 2 \leqslant q < \infty$ be numbers such that $L_X^p(\mathbf{R}^n)$ and $L_Y^p(\mathbf{R}^n)$ have type $r$ and cotype $q$.*

*Let $t : \Xi \to \mathcal{L}(X, Y)$ be a function which satisfies the $R$-boundedness condition*

$$\sum_{m \in \mathbf{Z}^n} \sum_{\alpha,\tilde{\alpha}} \bigg[\log^{1/r-1/q}\big(2 + |m|\big) \\ \times \sum_{\theta,\tilde{\theta}>0} \big\|\mathcal{R}\big(\big\{t\big(\theta, \tilde{\theta}, \mathrm{cb}(D,\alpha), \tilde{\mathrm{c}}\mathrm{b}(D \dot{+} m, \tilde{\alpha})\big) \colon D \ni \xi\big\}\big)\big\|_\infty \\ + \log^{1-1/q}\big(2 + |m|\big) \sum_{\theta>0} \big\|\mathcal{R}\big(\big\{t\big(\theta, 0, \mathrm{cb}(D,\alpha), \tilde{\mathrm{c}}\mathrm{b}(D \dot{+} m, \tilde{\alpha})\big) \colon D \ni \xi\big\}\big)\big\|_\infty$$

$$+\log^{1/r}\bigl(2+|m|\bigr)\sum_{\tilde\theta>0}\bigl\|\mathcal{R}\bigl(\bigl\{t\bigl(0,\tilde\theta,\mathrm{cb}(D\dot{-}m,\alpha),\tilde{\mathrm{c}}\mathrm{b}(D,\tilde\alpha)\bigr):\ D\ni\xi\bigr\}\bigr)\bigr\|_\infty\Bigr]$$
$$<\infty, \tag{6.13}$$

*where the otherwise undefined values of $t$ are taken to be zero, the quantities inside the $\infty$-norms are understood as functions of $\xi\in\mathbf{R}^n$, and "$D\ni\xi$" means that $D$ runs over all dyadic cubes $D\in\mathcal{D}$ such that $\xi\in D$.*

*Moreover, let there be functions $\tilde w\in BMO_{\mathrm{dy},U}(\mathbf{R}^n)$ and $w\in BMO_{\mathrm{dy},V}(\mathbf{R}^n)$, where $U\hookrightarrow\mathcal{L}(X,Y)$ and $V\hookrightarrow\mathcal{L}(Y',X')$ are UMD $R$-spaces, such that*

$$\bigl\langle w,h_Q^\theta\bigr\rangle_b' = \sum_{\tilde Q\in\tilde{\mathcal{Q}}_{\mathrm{gen}(Q)}} t(\theta,0,Q,\tilde Q)\tilde\beta_{\tilde Q}^{1/2},$$
$$\bigl\langle \tilde w,\tilde h_{\tilde Q}^{\tilde\theta}\bigr\rangle_{\tilde b} = \sum_{Q\in\mathcal{Q}_{\mathrm{gen}(\tilde Q)}} t(0,\tilde\theta,Q,\tilde Q)\beta_Q^{1/2} \tag{6.14}$$

*whenever $0<\theta<d_Q$, $0<\tilde\theta<\tilde d_{\tilde Q}$.*

*Then there exists a unique operator*

$$T\in\mathcal{L}\bigl(L_X^p\bigl(\mathbf{R}^n\bigr),L_Y^p\bigl(\mathbf{R}^n\bigr)\bigr)$$

*such that*

$$\bigl\langle y'\otimes\tilde h_{\tilde Q}^{\tilde\theta},TM_b\bigl(x\otimes h_Q^\theta\bigr)\bigr\rangle_{\tilde b}=\bigl\langle y',t(\theta,\tilde\theta,Q,\tilde Q)x\bigr\rangle$$
$$\text{for all } x\in X,\ y'\in Y' \text{ and } (\theta,\tilde\theta,Q,\tilde Q)\in\varXi.$$

Note that (6.13) is implied by the slightly stronger but much simpler-to-state condition

$$\sum_{m\in\mathbf{Z}^n}\log^{\max(1/r,1/q')}\bigl(2+|m|\bigr)\sum_\kappa\mathcal{R}(\mathrm{range}\,t_{\kappa,m})<\infty, \tag{6.15}$$

where $t_{\kappa,m}:=t_{\kappa,m}^{01}+t_{\kappa,m}^{10}+t_{\kappa,m}^{11}$. It is likely that in most examples the simpler condition (6.15) is quite sufficient; nevertheless, we have kept the more complicated form (6.13) in the statement of Theorem 6.12 to record the minimal assumptions under which we were able to carry out our proof. In the two important special cases when either $t$ is scalar-valued, or $X$ and $Y$ are Hilbert spaces (more generally, of cotype 2 and type 2, respectively), hence the $R$-bound is equivalent to the norm bound, the $\infty$-norm over the $R$-bounds of $\xi$-dependent sets as in (6.13) can be replaced by $R$-bounds over the whole range of $t_{\kappa,m}$ any way.

## 7. Standard Calderón–Zygmund operators

Our final purpose is to demonstrate how the more conventional style $Tb$ Theorem 1.1 for Calderón–Zygmund operators with standard kernels can be deduced from the abstract version in Theorem 6.12.

For the verification of (6.15), we need to compute and estimate the $R$-boundedness properties of the "matrix elements"

$$t\big(\theta, \tilde{\theta}, \mathrm{cb}(D,\alpha), \tilde{\mathrm{c}}\mathrm{b}(D \dot{+} m, \tilde{\alpha})\big) = \mathfrak{t}\big(h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}\big)$$

(where $Q = \mathrm{cb}(D,\alpha)$, $\tilde{Q} = \tilde{\mathrm{c}}\mathrm{b}(D \dot{+} m, \tilde{\alpha})$). This is similar to the corresponding task in [7]; however, unlike there, we only need to know $\mathfrak{t}(h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}})$ for $Q$ and $\tilde{Q}$ of the same generation. Of course, $\mathfrak{t}$ as appearing in Theorem 1.1 is not a priori defined on the non-smooth functions $h_Q^\theta$, but we can still make sense of the matrix elements of our interest by appropriate limit procedures for which we refer to [7], there being no real difference in comparison to the scalar case.

**Lemma 7.1.** *Under the standard $R$-estimates of Theorem* 1.1*, we have for all* $\theta, \tilde{\theta}, \alpha, \tilde{\alpha}$ *and all* $0 \neq m \in \mathbf{Z}^n$

$$\mathcal{R}\big(\big\{t\big(\theta, \tilde{\theta}, \mathrm{cb}(D,\alpha), \tilde{\mathrm{c}}\mathrm{b}(D \dot{+} m, \tilde{\alpha})\big) \colon D \in \mathcal{D}\big\}\big) \leqslant C|m|^{-n} \log^{-2}\big(2 + |m|\big).$$

**Proof.** Let us first consider $|m| > n$, in which case $D$ and $D \dot{+} m$ are necessarily disjoint and $\mathfrak{t}(h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}})$ can be directly determined in terms of the kernel $k$. If $\theta \neq 0$, then

$$\begin{aligned}
\mathfrak{t}\big(h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}\big) &= \iint\limits_{\mathbf{R}^n \times \mathbf{R}^n} \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}(x)\tilde{b}(x)k(x,y)b(y)h_Q^\theta(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \iint\limits_{\mathbf{R}^n \times \mathbf{R}^n} \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}(x)\tilde{b}(x)\big[k(x,y) - k(x,z)\big]b(y)h_Q^\theta(y)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

for any $z \neq x$. If $D = 2^j([0,1[^n + \ell)$ (with $j \in \mathbf{Z}$, $\ell \in \mathbf{Z}^n$), we make the change of variables $x = 2^j(u + \ell + m)$, $y = 2^j(v + \ell)$, and we choose $z := 2^j\ell$. Since $\operatorname{supp} h_Q^\theta \subset Q \subset D$ and similarly for $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$, it suffices to integrate over $(u,v) \in [0,1]^n \times [0,1]^n$. Thus

$$\begin{aligned}
\mathfrak{t}\big(h_Q^\theta \otimes \tilde{h}_{\tilde{Q}}^{\tilde{\theta}}\big) = \iint\limits_{[0,1]^n \times [0,1]^n} & 2^{jn/2}\big(\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}\tilde{b}\big)\big(2^j(u + \ell + m)\big) \\
& \times 2^{jn}\big[k\big(2^j(u + \ell + m), 2^j(v + \ell)\big) - k\big(2^j(u + \ell + m), 2^j\ell\big)\big] \\
& \times 2^{jn/2}\big(bh_Q^\theta\big)\big(2^j(v + \ell)\big)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

The first and last factors in the integrand are bounded functions, uniformly in all variables displayed. If the middle factor is multiplied by

$$|u+m-v|^n \log^2\bigl(|u+m-v|/|v|\bigr),$$

it belongs to an $R$-bounded set uniformly in all variables, according to the third condition in the standard $R$-estimates. Thus we find that

$$\begin{aligned}
&\mathcal{R}\bigl(\bigl\{t\bigl(\theta,\tilde\theta,\mathrm{cb}(D,\alpha),\tilde{\mathrm{c}}\mathrm{b}(D\dotplus m,\tilde\alpha)\bigr):\ D\in\mathcal{D}\bigr\}\bigr)\\
&\quad\leqslant C\iint\limits_{[0,1]^n\times[0,1]^n}|u+m-v|^{-n}\log^{-2}\biggl(\frac{|u+m-v|}{|v|}\biggr)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad\leqslant C|m|^{-n}\log^{-2}|m|
\end{aligned}$$

for $|m|>n$ and $\theta\neq 0$. A similar estimate is obtained for $\tilde\theta\neq 0$ by exploiting the second condition in the standard $R$-estimates instead of the third.

For $0<|m|\leqslant n$, we still use the kernel representation and write

$$\begin{aligned}
\mathfrak{t}\bigl(h_Q^\theta\otimes\tilde h_{\tilde Q}^{\tilde\theta}\bigr)=&\iint\limits_{[0,1]^n\times[0,1]^n}2^{jn/2}\bigl(\tilde h_{\tilde Q}^{\tilde\theta}\tilde b\bigr)\bigl(2^j(u+\ell+m)\bigr)|u+m-v|^{-n}\\
&\times\bigl|2^j(u+\ell+m)-2^j(v+\ell)\bigr|^n k\bigl(2^j(u+\ell+m),2^j(v+\ell)\bigr)\\
&\times 2^{jn/2}\bigl(bh_Q^\theta\bigr)\bigl(2^j(v+\ell)\bigr)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

The first and last factors are bounded as before, the third belongs to an $R$-bounded set for all values of the variables, and for $0\neq m\in\mathbf{Z}^n$ we have $\iint_{[0,1]^n\times[0,1]^n}|u+m-v|^{-n}\,\mathrm{d}u\,\mathrm{d}v\leqslant C$, thus $\mathcal{R}(\{t(\theta,\tilde\theta,\mathrm{cb}(D,\alpha),\tilde{\mathrm{c}}\mathrm{b}(D\dotplus m,\tilde\alpha)):\ D\in\mathcal{D}\})\leqslant C$ for $0<|m|\leqslant n$. □

**Lemma 7.2.** *Under the standard R-estimates and the weak R-boundedness property of Theorem* 1.1, *we have for all* $\theta,\tilde\theta,\alpha,\tilde\alpha$

$$\mathcal{R}\bigl(\bigl\{t\bigl(\theta,\tilde\theta,\mathrm{cb}(D,\alpha),\tilde{\mathrm{c}}\mathrm{b}(D,\tilde\alpha)\bigr):\ D\in\mathcal{D}\bigr\}\bigr)\leqslant C.$$

**Proof.** Recall that $h_Q^\theta$ is a linear combination of the indicators $1_C$ of a finite number of dyadic subcubes of $Q$ with comparable size, and the coefficients are bounded in absolute value by (a constant times) $|Q|^{-1/2}$. A similar assertion holds, of course, for $\tilde h_{\tilde Q}^{\tilde\theta}$. Thus the task is reduced to making sense of and estimating $|C|^{-1}\mathfrak{t}(1_C\otimes 1_{\tilde C})$ for all dyadic subcubes $C$ and $\tilde C$ of $D$ satisfying $\ell(C)=\ell(\tilde C)=2^{-2N}\ell(D)$. When $C\neq\tilde C$, this is handled just like the case $0<|m|\leqslant 2$ above. Thus we are left with bounding $\mathcal{R}(\{|D|^{-1}\mathfrak{t}(1_D\otimes 1_D):\ D\in\mathcal{D}\})$.

Fix a positive test-function $\Psi\in\mathcal{D}(\mathbf{R}^n)$ which is equal to 1 on $[0,1]^n$ and vanishes outside of $[-1,2]^n$. If $D=2^j([0,1[^n+\ell)$, denote $\Psi_D(x):=\Psi(2^{-j}x-\ell)$. Estimating with the kernel representation like in the case $0<|m|\leqslant 2$, we find that $\mathcal{R}(\{|D|^{-1}\mathfrak{t}(1_D\otimes$

$(1_D - \Psi_D))\colon\ D \in \mathcal{D}\}) \leqslant C$. Thus the problem is reduced to investigating $\mathcal{R}(\{|D|^{-1}\mathfrak{t}(1_D \otimes \Psi_D)\colon\ D \in \mathcal{D}\}) = \mathcal{R}(\{\mathfrak{t}(\mathcal{A}_\ell^{2^{-j}} 1_{[0,1[^n} \otimes \mathcal{A}_\ell^{2^{-j}} \Psi)\colon\ j \in \mathbf{Z}, \ell \in \mathbf{Z}^n\})$. (Recall the notion $\mathcal{A}_h^t f := t^{n/2} f(t \cdot - h)$.)

This is almost like what we have in the weak $R$-boundedness condition, except for the fact that $1_{[0,1[^n} \notin \mathcal{D}(\mathbf{R}^n)$. However, following [7, p. 41], we can expand $1_{[0,1[^n}$ as $\sum_{\mu=0}^{\infty} \sum_{\lambda \in I(\mu)} g_{\mu,\lambda}$, where $g_{\mu,\lambda} \in \mathcal{D}(\mathbf{R}^n)$ is supported in a ball $B(x_{\mu,\lambda}, 2^{-\mu})$ and satisfies $\|D^\alpha g_{\mu,\lambda}\|_\infty \leqslant C_\alpha 2^{\mu|\alpha|}$ uniformly in $\mu$ and $\lambda$; moreover, $|I(\mu)| \leqslant C 2^{\mu(n-1)}$. (This last estimate is related to the fact that the singular support of $1_{[0,1[^n}$ is of dimension $n-1$.) Corresponding to every $g_{\mu,\lambda}$ we split $\Phi = \sum_{i=1}^{2} \Phi_{\mu,\lambda}^i$, where $\Phi_{\mu,\lambda}^1 \in \mathcal{D}(\mathbf{R}^n)$ is supported in $B(x_{\mu,\lambda}, 8 \cdot 2^{-\mu})$ while $\Phi_{\mu,\lambda}^2 \in \mathcal{D}(\mathbf{R}^n)$ vanishes in $B(x_{\mu,\lambda}, 4 \cdot 2^{-\mu})$, and, moreover, $\|D^\alpha \Phi_{\mu,\lambda}^1\|_\infty \leqslant C_\alpha 2^{\mu|\alpha|}$, $\|\Phi_{\mu,\lambda}^2\|_\infty \leqslant C$.

Our purpose then is to show the convergence of

$$\sum_{\mu=0}^{\infty} \sum_{\lambda \in I(\mu)} \sum_{i=1}^{2} \mathcal{R}\big(\big\{\mathfrak{t}\big(\mathcal{A}_\ell^{2^{-j}} g_{\mu,\lambda}, \mathcal{A}_\ell^{2^{-j}} \Phi_{\mu,\lambda}^i\big)\colon\ j \in \mathbf{Z}, \ell \in \mathbf{Z}^n\big\}\big). \tag{7.3}$$

The terms with $i = 2$ are handled by kernel estimates similar to those in the cases $|m| > n$ and $0 < |m| \leqslant n$ above. By using the support properties and uniform boundedness of $g_{\mu,\lambda}$ and $\Phi_{\mu,\lambda}^i$, and writing the latter as a sum of $\Phi_{\mu,\lambda}^i 1_C$ where the $C$'s are cubes of side length $2^{-\mu}$, these yield

$$\begin{aligned}
&\mathcal{R}\big(\big\{\mathfrak{t}\big(\mathcal{A}_\ell^{2^{-j}} g_{\mu,\lambda}, \mathcal{A}_\ell^{2^{-j}} \Phi_{\mu,\lambda}^2\big)\colon\ j \in \mathbf{Z}, \ell \in \mathbf{Z}^n\big\}\big) \\
&\quad \leqslant C 2^{-\mu n} \sum_{0 \neq m \in \mathbf{Z}^n} |m|^{-n} \log^{-2}\big(2 + |m|\big) \leqslant C 2^{-\mu n}.
\end{aligned}$$

As for $i = 1$, we notice that $\tilde{g}_{\mu,\lambda}(x) := g_{\mu,\lambda}(2^{-\mu} x + x_{\mu,\lambda})$ and the similarly defined $\tilde{\Phi}_{\mu,\lambda}^1$ belong to a bounded subset of $\mathcal{D}(\mathbf{R}^n)$ for all $\mu$ and $\lambda$. Moreover, $g_{\mu,\lambda} = 2^{-\mu n/2} \mathcal{A}_{2^\mu x_{\mu,\lambda}}^{2^\mu} \tilde{g}_{\mu,\lambda}$, and then $\mathcal{A}_\ell^{2^{-j}} g_{\mu,\lambda} = 2^{-\mu n/2} \mathcal{A}_{2^\mu(\ell + x_{\mu,\lambda})}^{2^{\mu-j}} \tilde{g}_{\mu,\lambda}$ with a similar formula for $\Phi_{\lambda,\mu}^1$. Thus the weak $R$-boundedness property implies that

$$\mathcal{R}\big(\big\{\mathfrak{t}\big(\mathcal{A}_\ell^{2^{-j}} g_{\mu,\lambda}, \mathcal{A}_\ell^{2^{-j}} \Phi_{\mu,\lambda}^1\big)\colon\ j \in \mathbf{Z}, \ell \in \mathbf{Z}^n\big\}\big) \leqslant C 2^{-\mu n/2} \cdot 2^{-\mu n/2}.$$

Inserting these estimates and $|I(\mu)| \leqslant C 2^{n(\mu-1)}$, we find that (7.3) is bounded by $\sum_{\mu=0}^{\infty} |I(\mu)| \cdot C 2^{-\mu n} \leqslant C \sum_{\mu=0}^{\infty} 2^{-\mu} = C$. □

**Corollary 7.4.** *Under the standard $R$-estimates and the weak $R$-boundedness property assumptions of Theorem* 1.1, *condition* (6.15) *holds.*

**Proof.** From the two lemmata above we have

$$\mathcal{R}(\operatorname{range} t_{\kappa,m}) \leqslant C\big(1 + |m|\big)^{-n} \log^{-2}\big(2 + |m|\big).$$

It suffices to observe the summability of the series

$$\sum_{m\in\mathbf{Z}^n}\big(1+|m|\big)^{-n}\log^{\max(1/r,1/q')-2}\big(2+|m|\big)$$

since UMD spaces have type $r>1$ and cotype $q<\infty$ so that $\max(1/r,\,1/q')-2<-1$. $\square$

**Lemma 7.5.** *Under the assumptions of Theorem* 1.1*, conditions* (6.14) *hold with $w$ and $\tilde{w}$ as in Theorem* 1.1.

**Proof.** Let us fix a $\tilde{\theta}\neq 0$ and $\tilde{Q}\in\tilde{\mathcal{Q}}$, and consider the second summation in (6.14). We denote $D:=\mathrm{dy}(\tilde{Q})$.

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}_{\mathrm{gen}(\tilde{Q})}} t(0,\tilde{\theta},Q,\tilde{Q})\beta_Q^{1/2}\\
&\quad=\Bigg(\sum_{Q\in\mathcal{Q}_{\mathrm{gen}(\tilde{Q})},\ \mathrm{dy}(Q)=D}+\sum_{Q\in\mathcal{Q}_{\mathrm{gen}(\tilde{Q})},\ \mathrm{dy}(Q)\neq D}\Bigg)\mathfrak{t}\big(1_Q\otimes\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}\big)\\
&\quad=\mathfrak{t}\big(1_D\otimes\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}\big)+\sum_{Q\in\mathcal{Q}_{\mathrm{gen}(\tilde{Q})},\ \mathrm{dy}(Q)\neq D}\ \iint_{\mathbf{R}^n\times\mathbf{R}^n}\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}(x)\tilde{b}(x)k(x,y)b(y)1_Q(y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\mathfrak{t}\big(1_D\otimes\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}\big)+\iint_{\mathbf{R}^n\times\mathbf{R}^n}\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}(x)\tilde{b}(x)\big[k(x,y)-k(z,y)\big]b(y)\big(1-1_D(y)\big)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\mathfrak{t}\big(\chi\otimes\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}\big)+\iint_{\mathbf{R}^n\times\mathbf{R}^n}\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}(x)\tilde{b}(x)\big[k(x,y)-k(z,y)\big]b(y)\big(1-\chi(y)\big)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

where $z\in D$ and $\chi\in\mathcal{D}(\mathbf{R}^n)$ is equal to unity in a neighbourhood $\mathcal{O}$ of $D$. For $\psi\in\mathcal{D}(\mathbf{R}^n)$, with support in $\mathcal{O}$, in place of $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$, the last line above would equal $\int\tilde{w}(x)\tilde{b}(x)\psi(x)\,\mathrm{d}x$. Writing $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$ as a linear combination of indicators of dyadic subcubes of $\tilde{Q}$ and invoking the series expansion from the previous lemma, we can use approximation to extend this equality for $\tilde{h}_{\tilde{Q}}^{\tilde{\theta}}$, too. This gives the second condition in (6.14). The first one is proved similarly. $\square$

Taken together, the computations of this section show that Theorem 6.12 implies Theorem 1.1.

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